of rank 0, we refer back to the discussion of abelian Lie groups above. We begin with the case of rank 1.

Here the kernel of the map $[\ ,\ ]: \wedge^2\mathfrak{g} \to \mathfrak{g}$ is two dimensional, which means that for some $X \in \mathfrak{g}$ it consists of all vectors of the form $X \wedge Y$ with Y ranging over all of $\mathfrak{g}$ (X here will just be the vector corresponding to the hyperplane $\ker([\ ,\ ]) \subset \wedge^2\mathfrak{g}$ under the natural (up to scalars) duality between a three-dimensional vector space and its exterior square). Completing X to a basis $\{X, Y, Z\}$ of $\mathfrak{g}$, we can write $\mathfrak{g}$ in the form

$$[X, Y] = [X, Z] = 0,$$
$$[Y, Z] = \alpha X + \beta Y + \gamma Z$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$. If either β or γ is nonzero, we may now rechoose our basis, replacing Y by a multiple of the linear combination $\alpha X + \beta Y + \gamma Z$ and either leaving Z alone (if $\beta \neq 0$) or replacing Z by Y (if $\gamma \neq 0$). We will then have

$$[X, Y] = [X, Z] = 0,$$
$$[Y, Z] = Y$$

from which we see that $\mathfrak{g}$ is just the product of the one-dimensional abelian Lie algebra $\mathbb{C}X$ with the non-abelian two-dimensional Lie algebra $\mathbb{C}Y \oplus \mathbb{C}Z$ described in the preceding discussion. We may thus ignore this case and assume that in fact we have $\beta = \gamma = 0$; replacing X by αX we then have the Lie algebra

$$[X, Y] = [X, Z] = 0,$$
$$[Y, Z] = X.$$

How do we find the Lie groups with this Lie algebra? As before, we need to start with a faithful representation of $\mathfrak{g}$, but here the adjoint representation is useless, since X is in its kernel. We can, however, arrive at a representation of $\mathfrak{g}$ by considering the equations defining $\mathfrak{g}$: we want to find a pair of endomorphisms Y and Z on some vector space that do not commute, but that do commute with their commutator $X = [Y, Z]$; thus,

$$Y(YZ - ZY) - (YZ - ZY)Y = Y^2Z - 2YZY + ZY^2 = 0$$

and similarly for $[Z, [Y, Z]]$. One simple way to find such a pair of endomorphisms is make all three terms Y^2Z, YZY, and Z^2Y in the above equation zero, e.g., by making Y and Z both have square zero, and to have $YZ = 0$ while $ZY \neq 0$. For example, on a three-dimensional vector space with basis e_1, e_2, and e_3 we could take Y to be the map carrying e_3 to e_2 and killing e_1 and e_2, and Z the map carrying e_2 to e_1 and killing e_1 and e_3; we then have $YZ = 0$ while ZY sends e_3 to e_1. We see then that $\mathfrak{g}$ *is just the Lie algebra* $\mathfrak{n}_3$ *of strictly upper-triangular* 3×3 *matrices*. When we exponentiate we arrive at the group

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, a, b, c \in \mathbb{C} \right\}$$

which is simply connected. Now the center of G is the subgroup

$$Z(G) = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b \in \mathbb{C} \right\} \cong \mathbb{C},$$

so the discrete subgroups of $Z(G)$ are just lattices Λ of rank 1 or 2; thus any connected group with Lie algebra $\mathfrak{g}$ is either G, $G/\mathbb{Z}$, or $G/(\mathbb{Z} \times \mathbb{Z})$—that is, an extension of $\mathbb{C} \times \mathbb{C}$ by either $\mathbb{C}$, $\mathbb{C}^*$, or a torus E.

Exercise 10.3. Show that G/Λ is determined up to isomorphism by the one-dimensional $Z(G)/\Lambda$.

A similar analysis holds in the real case: just as before, $\mathfrak{n}_3$ is the unique real Lie algebra of dimension three with commutator subalgebra of dimension one; its simply connected form is the group G of unipotent 3×3 matrices and (the center of this group being $\mathbb{R}$) the only other group with this Lie algebra is the quotient $H = G/\mathbb{Z}$.

Incidentally, the group H represents an interesting example of a group that cannot be realized as a matrix group, i.e., that admits no faithful finite-dimensional representations. One way to see this is to argue that in any irreducible finite-dimensional representation V the center S^1 of H, being compact and abelian, must be diagonalizable; and so under the corresponding representation of the Lie algebra $\mathfrak{g}$ the element X must be carried to a diagonalizable endomorphism of V. But now if $v \in V$ is any eigenvector for X with eigenvalue λ, we also have, arguing as in §9.2,

$$X(Y(v)) = Y(X(v)) = Y(\lambda v) = \lambda Y(v)$$

and similarly $X(Z(v)) = \lambda Z(v)$, i.e., both $Y(v)$ and $Z(v)$ are also eigenvectors for X with eigenvalue λ. Since Y and Z generate $\mathfrak{g}$ and the representation V is irreducible, it follows that X must act as a scalar multiple $\lambda \cdot I$ of the identity; but since $X = [Y, Z]$ is a commutator and so has trace 0, it follows that $\lambda = 0$.

Exercise 10.4*. Show that if G is a simply connected Lie group, and its Lie algebra is solvable, then G cannot contain any nontrivial compact subgroup (in particular, it contains no elements of finite order).

The group H does, however, have an important infinite-dimensional representation. This arises from the representation of the Lie algebra $\mathfrak{g}$ on the space V of $\mathscr{C}^\infty$ functions on the real line $\mathbb{R}$ with coordinate x, in which Y, Z, and X are the operators

$$Y: f \mapsto \pi i x \cdot f,$$

$$Z: f \mapsto \frac{df}{dx}$$

and $X = [Y, Z]$ is $-\pi i$ times the identity. Exponentiating, we see that e^{tY} acts on a function f by multiplying it by the function $(\cos tx + i \cdot \sin tx)$; e^{tZ} sends f to the function F_t where $F_t(x) = f(t + x)$, and e^{tX} sends f to the scalar multiple $e^{-\pi i t} \cdot f$.

§10.3. Dimension Three, Rank 2

In this case, write the commutator subalgebra $\mathscr{D}\mathfrak{g} \subset \mathfrak{g}$ as the span of two elements Y and Z. The commutator of Y and Z can then be written

$$[Y, Z] = \alpha Y + \beta Z.$$

But now the endomorphism $\mathrm{ad}(Y)$ of $\mathfrak{g}$ carries $\mathfrak{g}$ into $\mathscr{D}\mathfrak{g}$, kills Y, and sends Z to $\alpha Y + \beta Z$, and so has trace β; on the other hand, since $\mathrm{ad}(Y)$ is a commutator in $\mathrm{End}(\mathfrak{g})$, it must have trace 0. Thus, β, and similarly α, must be zero; i.e., the subalgebra $\mathscr{D}\mathfrak{g}$ must be abelian. It follows from this that for any element $X \in \mathfrak{g}$ not in $\mathscr{D}\mathfrak{g}$, the map

$$\mathrm{ad}(X): \mathscr{D}\mathfrak{g} \to \mathscr{D}\mathfrak{g}$$

must be an isomorphism. We may now distinguish two possibilities: either $\mathrm{ad}(X)$ is diagonalizible or it is not.

(Note that for the first time we see a case where the classification of the real Lie algebra will be more complicated than that of the complex: in the real case we will have to deal with the third possibility that $\mathrm{ad}(X)$ is diagonalizible over $\mathbb{C}$ but not over $\mathbb{R}$, i.e., that it has two complex conjugate eigenvalues. Though we have not seen it much in these low-dimensional examples, in fact it is generally the case that the real picture is substantially more complicated than the complex one, for essentially just this reason.)

Possibility A: $\mathrm{ad}(X)$ *is diagonalizable.* In this case it is natural to use as a basis for $\mathscr{D}\mathfrak{g}$ a pair of eigenvectors Y, Z for $\mathrm{ad}(X)$; and by multiplying X by a suitable scalar we can assume that one of the eigenvalues (both of which are nonzero) is 1. We thus have the equations for $\mathfrak{g}$

$$[X, Y] = Y, \qquad [X, Z] = \alpha Z, \qquad [Y, Z] = 0 \tag{10.5}$$

for some $\alpha \in \mathbb{C}^*$.

Exercise 10.6. Show that two Lie algebras $\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}$ corresponding to two different scalars in the structure equations (10.5) are isomorphic if and only if $\alpha = \alpha'$ or

$\alpha = 1/\alpha'$. Observe that we have for the first time a continuously varying family of nonisomorphic complex Lie algebras.

To find the groups with these Lie algebras we go to the adjoint representation, which here is faithful. Explicitly, $\mathrm{ad}(Y)$ carries X to $-Y$ and kills Y and Z; $\mathrm{ad}(Z)$ carries X to $-\alpha Z$ and also kills Y and Z; and $\mathrm{ad}(X)$ carries Y to itself, Z to αZ, and kills X. A general member $aX - bY - cZ$ of the Lie algebra is thus represented (with respect to the basis $\{Y, Z, X\}$ for $\mathfrak{g}$) by the matrix

$$\begin{pmatrix} a & 0 & b \\ 0 & \alpha a & \alpha c \\ 0 & 0 & 0 \end{pmatrix}.$$

Exponentiating, we find that a group with Lie algebra $\mathfrak{g}$ is

$$G = \left\{ \begin{pmatrix} e^t & 0 & u \\ 0 & e^{\alpha t} & v \\ 0 & 0 & 1 \end{pmatrix}, t, u, v \in \mathbb{C} \right\} \subset \mathrm{GL}_3\mathbb{C}.$$

Here we run across a very interesting circumstance. If the complex number α is not rational, then the exponential map from $\mathfrak{g}$ to G is one-to-one, and hence a homeomorphism; thus, in particular, G is simply connected. If, on the other hand, α is rational, G will have nontrivial fundamental group. To see this, observe that we always have an exact sequence of groups

$$1 \to B \to G \to A \to 1,$$

where

$$A = \left\{ \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{\alpha t} & 0 \\ 0 & 0 & 1 \end{pmatrix}, t \in \mathbb{C} \right\}$$

and

$$B = \left\{ \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}, u, v \in \mathbb{C} \right\} \cong \mathbb{C} \times \mathbb{C}.$$

Now when $\alpha \notin \mathbb{Q}$, the group $A \cong \mathbb{C}$ is simply connected; but when $\alpha \in \mathbb{Q}$—whatever its denominator—we have $A \cong \mathbb{C}^*$ and correspondingly $\pi_1(G) = \mathbb{Z}$.

Exercise 10.7. Show that G has no center, and hence when $\alpha \neq \mathbb{Q}$, it is the unique connected group with Lie algebra $\mathfrak{g}$. For $\alpha \in \mathbb{Q}$, describe the universal covering of G and classify all groups with Lie algebra $\mathfrak{g}$.

Observe that in this case, even though we have a continuously varying family of Lie algebras $\mathfrak{g}_\alpha$, we have no corresponding continuously varying family of the adjoint (linear) Lie groups; the simply-connected forms do form a family, however.

Possibility B: $\mathrm{ad}(X)$ *is not diagonalizable.* In this case the natural thing to do is to choose a basis $\{Y, Z\}$ of $\mathscr{D}\mathfrak{g}$ with respect to which $\mathrm{ad}(X)$ is in Jordan normal form; replacing X by a multiple, we may assume both its eigenvalues are 1 so that we will have the Lie algebra

$$[X, Y] = Y, \qquad [X, Z] = Y + Z, \qquad [Y, Z] = 0. \tag{10.8}$$

With respect to the basis $\{Y, Z, X\}$ for $\mathfrak{g}$, then, the adjoint action of the general element $aX - bY - cZ$ of the Lie algebra is represented by the matrix

$$\begin{pmatrix} a & a & b+c \\ 0 & a & c \\ 0 & 0 & 0 \end{pmatrix}$$

and exponentiating we find that the corresponding group is

$$G = \left\{ \begin{pmatrix} e^t & te^t & u \\ 0 & e^t & v \\ 0 & 0 & 1 \end{pmatrix}, t, u, v \in \mathbb{C} \right\}.$$

Exercise 10.9. Show that this group has no center, and hence is the unique connected complex Lie group with its Lie algebra.

Note that the real Lie groups obtained by exponentiating the adjoint action of the Lie algebras given by (10.5) and (10.8) are all homeomorphic to $\mathbb{R}^3$ and have no center, and so are the only connected real Lie groups with these Lie algebras.

Exercise 10.10. Complete the analysis of real Lie groups in Case 2 by considering the third possibility mentioned above: that $\mathrm{ad}(X)$ acts on $\mathscr{D}\mathfrak{g}$ with distinct complex conjugate eigenvalues. Observe that in this way we arrive at our first example of two nonisomorphic real Lie algebras whose tensor products with $\mathbb{C}$ are isomorphic.

§10.4. Dimension Three, Rank 3

Our analysis of this final case begins, as in the preceding one, by looking for eigenvectors of the adjoint action of a suitable element $X \in \mathfrak{g}$. Specifically, we claim that we can find an element $H \in \mathfrak{g}$ such that $\mathrm{ad}(H)\colon \mathfrak{g} \to \mathfrak{g}$ has an eigenvector with nonzero eigenvalue. To see this, observe first that for any nonzero $X \in \mathfrak{g}$, the rank of $\mathrm{ad}(X)$ must be 2; in particular, we must have $\mathrm{Ker}(\mathrm{ad}(X)) = \mathbb{C}X$. Now start with any $X \in \mathfrak{g}$. Either $\mathrm{ad}(X)$ has an eigenvector with nonzero eigenvalue or it is nilpotent; if it is nilpotent, then there exists a vector $Y \in \mathfrak{g}$, not in the kernel of $\mathrm{ad}(X)$ but in the kernel of $\mathrm{ad}(X)^2$—that is, such that $\mathrm{ad}(X)(Y) = \alpha X$ for some nonzero $\alpha \in \mathbb{C}$. But then of course $\mathrm{ad}(Y)(X) = -\alpha X$, so that X is an eigenvector for $\mathrm{ad}(Y)$ with nonzero eigenvalue.

So: choose H and $X \in \mathfrak{g}$ so that X is an eigenvector with nonzero eigenvalue for $\mathrm{ad}(H)$, and write $[H, X] = \alpha X$. Since $H \in \mathscr{D}\mathfrak{g}$, $\mathrm{ad}(H)$ is a commutator in $\mathrm{End}(\mathfrak{g})$, and so has trace 0; it follows that $\mathrm{ad}(H)$ must have a third eigenvector Y with eigenvalue $-\alpha$. To describe the structure of $\mathfrak{g}$ completely it now remains to find the commutator of X and Y; but this follows from the Jacobi identity. We have

$$\begin{aligned}[H, [X, Y]] &= -[X, [Y, H]] - [Y, [H, X]] \\ &= -[X, \alpha Y] - [Y, \alpha X] \\ &= 0,\end{aligned}$$

from which we deduce that $[X, Y]$ must be a multiple of H; since it must be a nonzero multiple, we can multiply X or Y by a scalar to make it 1. Similarly multiplying H by a scalar we can assume α is 1 or any other nonzero scalar. Thus, there is only one possible complex Lie algebra $\mathfrak{g}$ of this type. One could look for endomorphisms H, X, and Y whose commutators satisfy these relations, as we did before. Or we may simply realize that the three-dimensional Lie algebra $\mathfrak{sl}_2\mathbb{C}$ has not yet been seen, so it must be this last possibility. In fact, a natural basis for $\mathfrak{sl}_2\mathbb{C}$ is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

whose Lie algebra is given by

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H. \tag{10.11}$$

What groups other than $\mathrm{SL}_2\mathbb{C}$ have Lie algebra $\mathfrak{sl}_2\mathbb{C}$? To begin with, the group $\mathrm{SL}_2\mathbb{C}$ is simply connected: for example, the map $\mathrm{SL}_2\mathbb{C} \to \mathbb{C}^2 - \{(0, 0)\}$ sending a matrix to its first row expresses the topological space $\mathrm{SL}_2\mathbb{C}$ as a bundle with fiber $\mathbb{C}$ over $\mathbb{C}^2 - \{(0, 0)\}$. Also, it is not hard to see that the center of $\mathrm{SL}_2\mathbb{C}$ is just the subgroup $\{\pm I\}$ of scalar matrices, so that the only other connected group with Lie algebra $\mathfrak{sl}_2\mathbb{C}$ is the quotient $\mathrm{PSL}_2\mathbb{C} = \mathrm{Sl}_2\mathbb{C}/\{\pm I\}$.

As in the preceding case, the analysis of real three-dimensional Lie algebras $\mathfrak{g}$ with $\mathscr{D}\mathfrak{g} = \mathfrak{g}$ involves one additional possibility. At the outset of the argument above, we started with an arbitrary $H \in \mathfrak{g}$ and said that if $\mathrm{ad}(H)$ had no eigenvector other than H itself, then it would have to be nilpotent. Of course, in the real case it is also possible that $\mathrm{ad}(H)$ has two distinct complex conjugate eigenvalues λ and $\bar{\lambda}$. Since $\mathrm{ad}(H)$ is a commutator in $\mathrm{End}(\mathfrak{g})$ and so has trace 0, λ will have to be purely imaginary in this case; and so multiplying H by a real scalar we can assume that its eigenvalues are i and $-i$. It follows then that we can find $X, Y \in \mathfrak{g}$ with

$$[H, X] = Y \quad \text{and} \quad [H, Y] = -X.$$

Using the Jacobi identity as before we may conclude that the commutator of X and Y is a multiple of H; after multiplying each of X and Y by a real scalar we can assume that it is either H or $-H$. Finally, if $[X, Y] = -H$, then we observe that we are in the case we considered before: $\operatorname{ad}(Y)$ will have $X + H$ as an eigenvector with nonzero eigenvalue, and following our previous analysis we may conclude that $\mathfrak{g} \cong \mathfrak{sl}_2\mathbb{R}$. Thus, we are left with the sole additional possibility that $\mathfrak{g}$ has structure equations

$$[H, X] = Y, \qquad [H, Y] = -X, \qquad [X, Y] = H. \tag{10.12}$$

This, finally, we may recognize as the Lie algebra $\mathfrak{su}_2$ of the real Lie group $SU(2)$ (as you may recall, the isomorphism $\mathfrak{su}_2 \otimes \mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$ was used in the last lecture).

What are the real Lie groups with Lie algebras $\mathfrak{sl}_2\mathbb{R}$ and $\mathfrak{su}_2$? To start, the center of the group $SL_2\mathbb{R}$ is again just the scalar matrices $\{\pm I\}$, so the only group dominated by $SL_2\mathbb{R}$ is the quotient $PSL_2\mathbb{R}$. On the other hand, unlike the complex case $SL_2\mathbb{R}$ is not simply connected: now the map associating to a 2×2 matrix its first row expresses $SL_2\mathbb{R}$ as a bundle with fiber $\mathbb{R}$ over $\mathbb{R}^2 - \{(0, 0)\}$, so that $\pi_1(SL_2\mathbb{R}) = \mathbb{Z}$. More precisely $PSL_2\mathbb{R}$ maps to the real projective line $\mathbb{P}^1\mathbb{R}$, which is homeomorphic to the circle, with fiber homeomorphic to $\mathbb{R}^2$, so $\pi_1(PSL_2\mathbb{R}) = \mathbb{Z}$. We thus have a tower of covering spaces of $PSL_2\mathbb{R}$, consisting of the simply-connected group $\tilde{S}$ with center $\mathbb{Z}$ and its quotients $\tilde{S}_n = \tilde{S}/n\mathbb{Z}$ (not all of these are covers of $SL_2\mathbb{R}$, despite the diagram below).

A note: In §10.2 we encountered a real Lie group with no faithful finite-dimensional representations; only its universal cover could be represented as a matrix group. Here we find in some sense the opposite phenomenon: the groups $\tilde{S}$ and $\tilde{S}_n$ have no faithful finite-dimensional representations, all finite-dimensional representations factoring through $SL_2\mathbb{R}$ or $PSL_2\mathbb{R}$. This fact will be proved as a consequence of our discussion of the representations of the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ in the next lecture.

What about groups with Lie algebra $\mathfrak{su}_2$? To begin with, there is $SU(2)$, which (again via the map sending a matrix to its first row vector) is homeomorphic to S^3 and thus simply connected. The center of this group is again $\{\pm I\}$, so that the quotient $PSU(2)$ is the only other group with Lie algebra $\mathfrak{su}_2$. (Alternatively, we may realize $SU(2)$ as the group of unit quaternions, cf. Exercise 7.15.)

Finally, we remark that there are other representations of the real and complex Lie groups discussed above. As we will see, the Lie algebra $\mathfrak{so}_3\mathbb{C}$ is isomorphic to $\mathfrak{sl}_2\mathbb{C}$, which induces an isomorphism between the corresponding adjoint forms $PSL_2\mathbb{C}$ and $SO_3\mathbb{C}$ (and between the simply-connected forms $SL_2\mathbb{C}$ and the spin group $\mathrm{Spin}_3\mathbb{C}$). This in turn suggests two more real forms of this group: $SO_3\mathbb{R}$ and $SO^+(2, 1)$. In fact, it is not hard to see that $SO_3\mathbb{R} \cong PSU(2)$, while $SO^+(2, 1) \cong PSL_2\mathbb{R}$. Lastly the isomorphism $\mathfrak{su}_{1,1} \otimes \mathbb{C} \cong$

$\mathfrak{su}_2 \otimes \mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$ implies that the real Lie algebra $\mathfrak{su}_{1,1}$ is isomorphic to either $\mathfrak{su}_2$ or $\mathfrak{sl}_2\mathbb{R}$; in fact, the latter is the case and this induces an isomorphism of groups $\mathrm{SU}_{1,1} \cong \mathrm{SL}_2\mathbb{R}$. We summarize the isomorphisms mentioned in the diagram below:

$$\begin{tikzcd}
\tilde{S} \arrow[d] & & & \\
\tilde{S}_n \arrow[d] & \mathrm{Spin}_3\mathbb{C} \arrow[d, equal] & \mathrm{Spin}_3\mathbb{R} \arrow[d, equal] & \\
\mathrm{SU}(1,1) = \mathrm{SL}_2\mathbb{R} \arrow[r, hook] \arrow[d] & \mathrm{SL}_2\mathbb{C} \arrow[d] & \mathrm{SU}(2) \arrow[l, hook'] \arrow[d] & = \{\text{unit quaternions}\} \\
\mathrm{PSL}_2\mathbb{R} \arrow[r, hook] \arrow[d, equal] & \mathrm{PSL}_2\mathbb{C} \arrow[d, equal] & \mathrm{PSU}(2) \arrow[l, hook'] \arrow[d, equal] & = \{\text{unit quaternions}\}/\pm 1 \\
\mathrm{SO}^+(2,1) \arrow[r, hook] & \mathrm{SO}_3\mathbb{C} & \mathrm{SO}_3\mathbb{R} \arrow[l, hook'] &
\end{tikzcd} \tag{10.13}$$

Note also the coincidences:

$$\mathrm{Sp}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C}), \qquad \mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}), \tag{10.14}$$

which follow from the fact that Sp refers to preserving a skew-symmetric bilinear form, and for 2×2 matrices the determinant is such a form.

Exercise 10.15. Identify the Lie algebras $\mathfrak{so}_3$, $\mathfrak{su}_2$, $\mathfrak{su}_{1,1}$, $\mathfrak{so}_{2,1}$, and verify the assertions made about the corresponding Lie groups in the diagram.

Exercise 10.16. For each of the Lie algebras encountered in this lecture, compute the lower central series and the derived series, and say whether the algebra is nilpotent, solvable, simple, or semisimple.

Exercise 10.17. The following are Lie groups of dimension two or three, so must appear on our list. Find them: (i) the group of affine transformations of the line ($x \mapsto ax + b$, under composition); (ii) the group of upper-triangular 2×2 matrices; (iii) the group of orientation preserving Euclidean transformations of the plane (compositions of translations and rotations).

Exercise 10.18. Locate $\mathbb{R}^3$ with the usual cross-product on our list of Lie algebras. More generally, consider the family of Lie algebras parametrized by real quadruples (a, b, c, d), each with basis X, Y, Z with bracket given by

$$[X, Y] = aZ + dY, \qquad [Y, Z] = bX, \qquad [Z, X] = cY - dZ.$$

Classify this Lie algebra as (a, b, c, d) varies in $\mathbb{R}^4$, showing in particular that every three-dimensional Lie algebra can be written in this way.

Exercise 10.19. Realize the isomorphism of $SU(1, 1)$ with $SL_2\mathbb{R}$ by identifying them with the groups of complex automorphisms of the unit disk and the upper half-plane, respectively.

Exercise 10.20. Classify all Lie algebras of dimension four and rank 1; in particular, show that they are all direct sums of Lie algebras described above.

Exercise 10.21. Show more generally that there exists a Lie algebra of dimension m and rank 1 that is not a direct sum of smaller Lie algebras if and only if m is odd; in case m is odd show that this Lie algebra is unique and realize it as a Lie subalgebra of $\mathfrak{sl}_n\mathbb{C}$.

LECTURE 11

Representations of $\mathfrak{sl}_2\mathbb{C}$

This is the first of four lectures—§11–14—that comprise in some sense the heart of the book. In particular, the naive analysis of §11.1, together with the analogous parts of §12 and §13, form the paradigm for the study of finite-dimensional representations of all semisimple Lie algebras and groups. §11.2 is less central; in it we show how the analysis carried out in §11.1 can be used to explicitly describe the tensor products of irreducible representations. §11.3 is least important; it indicates how we can interpret geometrically some of the results of the preceding section. The discussions in §11.1 and §11.2 are completely elementary (we do use the notion of symmetric powers of a vector space, but in a non-threatening way). §11.3 involves a fair amount of classical projective geometry, and can be skimmed or skipped by those not already familiar with the relevant basic notions from algebraic geometry.

§11.1. The Irreducible Representations

We start our discussion of representations of semisimple Lie algebras with the simplest case, that of $\mathfrak{sl}_2\mathbb{C}$. As we will see, while this case does not exhibit any of the complexity of the more general case, the basic idea that informs the whole approach is clearly illustrated here.

This approach is one already mentioned above, in connection with the representations of the symmetric group on three letters. The idea in that case was that given a representation of our group on a vector space V we first restrict the representation to the abelian subgroup generated by a 3-cycle τ. We obtain a decomposition

$$V = \bigoplus V_\alpha$$

of V into eigenspaces for the action of τ; the commutation relations satisfied by the remaining elements σ of the group with respect to τ implied that such σ simply permuted these subspaces V_α, so that the representation was in effect determined by the collection of eigenvalues of τ.

Of course, circumstances in the case of Lie algebra representations are quite different: to name two, it is no longer the case that the action of an abelian object on any vector space admits such a decomposition; and even if such a decomposition exists we certainly cannot expect that the remaining elements of our Lie algebra will simply permute its summands. Nevertheless, the idea remains essentially a good one, as we shall now see.

To begin with, we choose the basis for the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ from the last lecture:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfying

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H. \tag{11.1}$$

These seem like a perfectly natural basis to choose, but in fact the choice is dictated by more than aesthetics; there is, as we will see, a nearly canonical way of choosing a basis of a semisimple Lie algebra (up to conjugation), which will yield this basis in the present circumstance and which will share many of the properties we describe below.

In any event, let V be an irreducible finite-dimensional representation of $\mathfrak{sl}_2\mathbb{C}$. We start by trotting out one of the facts that we quoted in Lecture 9, the preservation of Jordan decomposition; in the present circumstances it implies that

The action of H on V is diagonalizable. (11.2)

We thus have, as indicated, a decomposition

$$V = \bigoplus V_\alpha, \tag{11.3}$$

where the α run over a collection of complex numbers, such that for any vector $v \in V_\alpha$ we have

$$H(v) = \alpha \cdot v.$$

The next question is obviously how X and Y act on the various spaces V_α. We claim that X and Y must each carry the subspaces V_α into other subspaces $V_{\alpha'}$. In fact, we can be more specific: if we want to know where the image of a given vector $v \in V_\alpha$ under the action of X sits in relation to the decomposition (11.3), we have to know how H acts on $X(v)$; this is given by the

Fundamental Calculation (first time):

$$\begin{aligned} H(X(v)) &= X(H(v)) + [H, X](v) \\ &= X(\alpha \cdot v) + 2X(v) \\ &= (\alpha + 2) \cdot X(v); \end{aligned}$$

i.e., *if v is an eigenvector for H with eigenvalue α, then $X(v)$ is also an eigenvector for H, with eigenvalue $\alpha + 2$.* In other words, we have

$$X \colon V_\alpha \to V_{\alpha+2}.$$

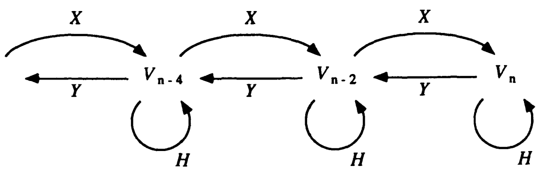
The action of Y on each V_α is similarly calculated; we have $Y(V_\alpha) \subset V_{\alpha-2}$.

Observe that as an immediate consequence of this and the irreducibility of V, *all the complex numbers α that appear in the decomposition* (11.3) *must be congruent to one another* mod 2: for any α_0 that actually occurs, the subspace

$$\bigoplus_{n \in \mathbb{Z}} V_{\alpha_0 + 2n}$$

would be invariant under $\mathfrak{sl}_2\mathbb{C}$ and hence equal to all of V. Moreover, by the same token, the V_α that appear must form an unbroken string of numbers of the form $\beta, \beta + 2, \ldots, \beta + 2k$. We denote by n the last element in this sequence; at this point we just know n is a complex number, but we will soon see that it must be an integer.

To proceed with our analysis, we have the following picture of the action of $\mathfrak{sl}_2\mathbb{C}$ on the vector space V:



Choose any nonzero vector $v \in V_n$; since $V_{n+2} = (0)$, we must have $X(v) = 0$. We ask now what happens when we apply the map Y to the vector v. To begin with, we have

Claim 11.4. *The vectors $\{v, Y(v), Y^2(v), \ldots\}$ span V.*

Proof. From the irreducibility of V it is enough to show that the subspace $W \subset V$ spanned by these vectors is carried into itself under the action of $\mathfrak{sl}_2\mathbb{C}$. Clearly, Y preserves W, since it simply carries the vector $Y^m(v)$ into $Y^{m+1}(v)$. Likewise, since the vector $Y^m(v)$ is in V_{n-2m}, we have $H(Y^m(v)) = (n - 2m) \cdot Y^m(v)$, so H preserves the subspace W. Thus, it suffices to check that $X(W) \subset W$, i.e., that for each m, X carries $Y^m(v)$ into a linear combination of the $Y^i(v)$. We check this in turn for $m = 0, 1, 2$, etc.

To begin with, we have $X(v) = 0 \in W$. To see what X does to $Y(v)$, we use the commutation relations for $\mathfrak{sl}_2\mathbb{C}$: we have

$$\begin{aligned} X(Y(v)) &= [X, Y](v) + Y(X(v)) \\ &= H(v) + Y(0) \\ &= n \cdot v. \end{aligned}$$

Next, we see that

$$\begin{aligned} X(Y^2(v)) &= [X, Y](Y(v)) + Y(X(Y(v))) \\ &= H(Y(v)) + Y(n \cdot v) \\ &= (n-2) \cdot Y(v) + n \cdot Y(v). \end{aligned}$$

The pattern now is clear: X carries each vector in the sequence v, $Y(v)$, $Y^2(v)$, ... into a multiple of the previous vector. Explicitly, we have

$$X(Y^m(v)) = (n + (n-2) + (n-4) + \cdots + (n - 2m + 2)) \cdot Y^{m-1}(v),$$

or

$$X(Y^m(v)) = m(n - m + 1) \cdot Y^{m-1}(v), \tag{11.5}$$

as can readily be verified by induction. □

There are a number of corollaries of the calculation in the above Claim. To begin with, we make the observation that

$$\textit{all the eigenspaces } V_\alpha \textit{ of } H \textit{ are one dimensional.} \tag{11.6}$$

Second, since we have in the course of the proof written down a basis for V and said exactly where each of H, X, and Y takes each basis vector, the representation V is completely determined by the one complex number n that we started with; in particular, of course, we have that

V is determined by the collection of α occurring in the decomposition $V = \bigoplus V_\alpha$. (11.7)

To complete our analysis, we have to use one more time the finite dimensionality of V. This tells us that there is a lower bound on the α for which $V_\alpha \neq (0)$ as well as an upper one, so that we must have $Y^k(v) = 0$ for sufficiently large k. But now if m is the smallest power of Y annihilating v, then from the relation (11.5),

$$0 = X(Y^m(v)) = m(n - m + 1) \cdot Y^{m-1}(v),$$

and the fact that $Y^{m-1}(v) \neq 0$, we conclude that $n - m + 1 = 0$; in particular, it follows that *n is a non-negative integer*. The picture is thus that the eigenvalues α of H on V form a string of integers differing by 2 and symmetric about the origin in $\mathbb{Z}$. In sum, then, we see that there is a unique representation $V^{(n)}$ for each non-negative integer n; the representation *$V^{(n)}$ is $(n+1)$-dimensional, with H having eigenvalues $n, n-2, \ldots, -n+2, -n$.*

Note that the existence part of this statement may be deduced by checking that the actions of H, X, and Y as given above in terms of the basis v, Yv, $Y^2(v), \ldots, Y^n(v)$ for V do indeed satisfy all the commutation relations for $\mathfrak{sl}_2\mathbb{C}$. Alternatively, we will exhibit them in a moment. Note also that by the symmetry of the eigenvalues we may deduce the useful fact that *any representation V of $\mathfrak{sl}_2\mathbb{C}$ such that the eigenvalues of H all have the same parity and occur with multiplicity one is necessarily irreducible*; more generally, *the number of irreducible factors in an arbitrary representation V of $\mathfrak{sl}_2\mathbb{C}$ is exactly the sum of the multiplicities of* 0 *and* 1 *as eigenvalues of H.*

We can identify in these terms some of the standard representations of $\mathfrak{sl}_2\mathbb{C}$. To begin with, the trivial one-dimensional representation $\mathbb{C}$ is clearly just $V^{(0)}$. As for the standard representation of $\mathfrak{sl}_2\mathbb{C}$ on $V = \mathbb{C}^2$, if x and y are the standard basis for $\mathbb{C}^2$, then we have $H(x) = x$ and $H(y) = -y$, so that $V = \mathbb{C}\cdot x \oplus \mathbb{C}\cdot y = V_{-1} \oplus V_1$ is just the representation $V^{(1)}$ above. Similarly, a basis for the symmetric square $W = \mathrm{Sym}^2 V = \mathrm{Sym}^2\mathbb{C}^2$ is given by $\{x^2, xy, y^2\}$, and we have

$$\begin{aligned} H(x\cdot x) &= x\cdot H(x) + H(x)\cdot x = 2x\cdot x, \\ H(x\cdot y) &= x\cdot H(y) + H(x)\cdot y = 0, \\ H(y\cdot y) &= y\cdot H(y) + H(y)\cdot y = -2y\cdot y, \end{aligned}$$

so the representation $W = \mathbb{C}\cdot x^2 \oplus \mathbb{C}\cdot xy \oplus \mathbb{C}\cdot y^2 = W_{-2} \oplus W_0 \oplus W_2$ is the representation $V^{(2)}$ above. More generally, the nth symmetric power $\mathrm{Sym}^n V$ of V has basis $\{x^n, x^{n-1}y, \ldots, y^n\}$, and we have

$$\begin{aligned} H(x^{n-k}y^k) &= (n-k)\cdot H(x)\cdot x^{n-k-1}y^k + k\cdot H(y)\cdot x^{n-k}y^{k-1} \\ &= (n-2k)\cdot x^{n-k}y^k \end{aligned}$$

so that the eigenvalues of H on $\mathrm{Sym}^n V$ are exactly $n, n-2, \ldots, -n$. By the observation above that a representation for which all eigenvalues of H occur with multiplicity 1 must be irreducible, it follows that $\mathrm{Sym}^n V$ is irreducible, and hence that

$$V^{(n)} = \mathrm{Sym}^n V.$$

In sum then, we can say simply that

Any irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ is a symmetric power of the standard representation $V \cong \mathbb{C}^2$. (11.8)

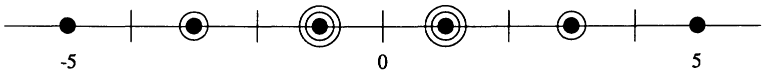
Observe that when we exponentiate the image of $\mathfrak{sl}_2\mathbb{C}$ under the embedding $\mathfrak{sl}_2\mathbb{C} \to \mathfrak{sl}_{n+1}\mathbb{C}$ corresponding to the representation $\mathrm{Sym}^n V$, we arrive at the group $\mathrm{SL}_2\mathbb{C}$ when n is odd and $\mathrm{PGL}_2\mathbb{C}$ when n is even. Thus, *the representations of the group* $\mathrm{PGL}_2\mathbb{C}$ *are exactly the even powers* $\mathrm{Sym}^{2n}V$.

Exercise 11.9. Use the analysis of the representations of $\mathfrak{sl}_2\mathbb{C}$ to prove the statement made in the previous lecture that the universal cover $\widetilde{S}$ of $\mathrm{SL}_2\mathbb{R}$ has no finite-dimensional representations.

§11.2. A Little Plethysm

Clearly, knowing the eigenspace decomposition of given representations tells us the eigenspace decomposition of all their tensor, symmetric, and alternating products and powers: for example, if $V = \bigoplus V_\alpha$ and $W = \bigoplus W_\beta$ then $V \otimes W = \bigoplus (V_\alpha \otimes W_\beta)$ and $V_\alpha \otimes W_\beta$ is an eigenspace for H with eigenvalue $\alpha + \beta$. We can use this to describe the decomposition of these products and powers into irreducible representations of the algebra $\mathfrak{sl}_2\mathbb{C}$.

For example, let $V \cong \mathbb{C}^2$ be the standard representation of $\mathfrak{sl}_2\mathbb{C}$; and suppose we want to study the representation $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^3 V$; we ask in particular whether if it irreducible and, if not, how it decomposes. We have seen that the eigenvalues of $\mathrm{Sym}^2 V$ are 2, 0, and -2, and those of $\mathrm{Sym}^3 V$ are 3, 1, -1, and -3. The 12 eigenvalues of the tensor product $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^3 V$ are thus 5 and -5, 3 and -3 (taken twice), and 1 and -1 (taken three times); we may represent them by the diagram



The eigenvector with eigenvalue 5 will generate a subrepresentation of the tensor product isomorphic to $\mathrm{Sym}^5 V$, which will account for one occurrence of each of the eigenvalues 5, 3, 1, -1, -3, and -5. Similarly, the complement of $\mathrm{Sym}^5 V$ in the tensor product will have eigenvalues 3 and -3, and 1 and -1 (taken twice), and so will contain a copy of the representation $\mathrm{Sym}^3 V$, which will account for one occurrence of the eigenvalues 3, 1, -1 and -3; and the complement of these two subrepresentations will be simply a copy of V. We have, thus,

$$\mathrm{Sym}^2 V \otimes \mathrm{Sym}^3 V \cong \mathrm{Sym}^5 V \oplus \mathrm{Sym}^3 V \oplus V.$$

Note that the projection map

$$\mathrm{Sym}^2 V \otimes \mathrm{Sym}^3 V \to \mathrm{Sym}^5 V$$

on the first factor is just multiplication of polynomials; the other two projections do not admit such obvious interpretations.

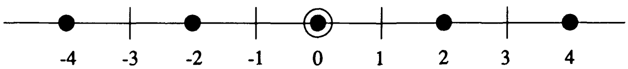
Exercise 11.10. Find, in a similar way, the decomposition of the tensor product $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^5 V$.

Exercise 11.11*. Show, in general, that for $a \geq b$ we have

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V = \mathrm{Sym}^{a+b} V \oplus \mathrm{Sym}^{a+b-2} V \oplus \cdots \oplus \mathrm{Sym}^{a-b} V.$$

As indicated, we can also look at symmetric and exterior powers of given representations; in many ways this is more interesting. For example, let

$V \cong \mathbb{C}^2$ be as above the standard representation of $\mathfrak{sl}_2\mathbb{C}$, and let $W = \mathrm{Sym}^2 V$ be its symmetric square; i.e., in the notation introduced above, take $W = V^{(2)}$. We ask now whether the symmetric square of W is irreducible, and if not what its decomposition is. To answer this, observe that W has eigenvalues -2, 0, and 2, each occurring once, so that the symmetric square of W will have eigenvalues the pairwise sums of these numbers—that is, -4, -2, 0 (occurring twice), 2, and 4. We may represent $\mathrm{Sym}^2 V$ by the diagram:



From this, it is clear that the representation $\mathrm{Sym}^2 W$ must decompose into one copy of the representation $V^{(4)} = \mathrm{Sym}^4 V$, plus one copy of the trivial (one-dimensional) representation:

$$\mathrm{Sym}^2(\mathrm{Sym}^2 V)) = \mathrm{Sym}^4 V \oplus \mathrm{Sym}^0 V. \tag{11.12}$$

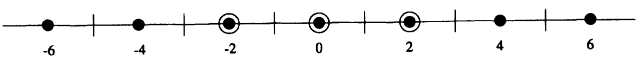
Indeed, we can see this directly: we have a natural map

$$\mathrm{Sym}^2(\mathrm{Sym}^2 V)) \to \mathrm{Sym}^4 V$$

obtained simply by evaluation; this will have a one-dimensional kernel (if x and y are as above the standard basis for V we can write a generator of this kernel as $(x^2)\cdot(y^2) - (x \cdot y)^2$).

Exercise 11.13. Show that the exterior square $\wedge^2 W$ is isomorphic to W itself. Observe that this, together with the above description of $\mathrm{Sym}^2 W$, agrees with the decomposition of $W \otimes W$ given in Exercise 11.11 above.

We can, in a similar way, describe the decomposition of all the symmetric powers of the representation $W = \mathrm{Sym}^2 V$. For example, the third symmetric power $\mathrm{Sym}^3 W$ has eigenvalues given by the triple sums of the set $\{-2, 0, 2\}$; these are -6, -4, -2 (twice), 0 (twice), 2 (twice), 4, and 6; diagrammatically,



Again, there is no ambiguity about the decomposition; this collection of eigenspaces can only come from the direct sum of $\mathrm{Sym}^6 V$ with $\mathrm{Sym}^2 V$, so we must have

$$\mathrm{Sym}^3(\mathrm{Sym}^2 V) = \mathrm{Sym}^6 V \oplus \mathrm{Sym}^2 V.$$

As before, we can see at least part of this directly: we have a natural evaluation map

$$\mathrm{Sym}^3(\mathrm{Sym}^2 V) \to \mathrm{Sym}^6 V,$$

and the eigenspace decomposition tells us that the kernel is the irreducible representation $\mathrm{Sym}^2 V$.

Exercise 11.14. Use the eigenspace decomposition to establish the formula

$$\mathrm{Sym}^n(\mathrm{Sym}^2 V) = \bigoplus_{\alpha=0}^{[n/2]} \mathrm{Sym}^{2n-4\alpha} V$$

for all n.

§11.3. A Little Geometric Plethysm

We want to give some geometric interpretations of these and similar decompositions of higher tensor powers of representations of $\mathfrak{sl}_2\mathbb{C}$. One big difference is that instead of looking at the action of either the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ or the groups $\mathrm{SL}_2\mathbb{C}$ or $\mathrm{PGL}_2\mathbb{C}$ on a representation W, we look at the action of the group $\mathrm{PGL}_2\mathbb{C}$ on the associated projective space[1] $\mathbb{P}W$. In this context, it is natural to look at various geometric objects associated to the action: for example, we look at closures of orbits of the action, which all turn out to be algebraic varieties, i.e., definable by polynomial equations. In particular, our goal in the following will be to describe the symmetric and exterior powers of W in terms of the action of $\mathrm{PGL}_2\mathbb{C}$ on the projective spaces $\mathbb{P}W$ and various loci in $\mathbb{P}W$.

The main point is that while the action of $\mathrm{PGL}_2\mathbb{C}$ on the projective space $\mathbb{P}V \cong \mathbb{P}^1$ associated to the standard representation V is transitive, its action on the spaces $\mathbb{P}(\mathrm{Sym}^n V) \cong \mathbb{P}^n$ for $n > 1$ is not. Rather, the action will preserve various orbits whose closures are algebraic subvarieties of $\mathbb{P}^n$—for example, the locus of points

$$C = \{[v \cdot v \cdot \ldots \cdot v] : v \in V\} \subset \mathbb{P}(\mathrm{Sym}^n V)$$

corresponding to nth powers in $\mathrm{Sym}^n V$ will be an algebraic curve in $\mathbb{P}(\mathrm{Sym}^n V) \cong \mathbb{P}^n$, called the *rational normal curve*; and this curve will be carried into itself by any element of $\mathrm{PGL}_2\mathbb{C}$ acting on $\mathbb{P}^n$ (more about this in a moment). Thus, a knowledge of the geometry of these subvarieties of $\mathbb{P}W$ may illuminate the representation W, and vice versa. This approach is particularly useful in describing the symmetric powers of W, since these powers can be viewed as the vector spaces of homogeneous polynomials on the projective space $\mathbb{P}(W^*)$ (or, mod scalars, as hypersurfaces in that projective space). Decomposing these symmetric powers should therefore correspond to some interesting projective geometry.

[1] $\mathbb{P}W$ here denotes the projective space of lines through the origin in W, or the quotient space of $W \setminus \{0\}$ by multiplication by nonzero scalars; we write $[w]$ for the point in $\mathbb{P}W$ determined by the nonzero vector w. For $W = \mathbb{C}^{m+1}$, $[z_0, \ldots, z_m]$ is the point in $\mathbb{P}^m = \mathbb{P}W$ determined by a point $(z_0, \ldots, z_m)$ in $\mathbb{C}^{m+1}$.

Digression on Projective Geometry

First, as we have indicated, we want to describe representations of Lie groups in terms of the corresponding actions on projective spaces. The following fact from algebraic geometry is therefore of some moral (if not logical) importance:

Fact 11.15. The group of automorphisms of projective space $\mathbb{P}^n$—either as algebraic variety or as complex manifold—is just the group $\mathrm{PGL}_{n+1}\mathbb{C}$.

For a proof, see [Ha]. (For the Riemann sphere $\mathbb{P}^1$ at least, this should be a familiar fact from complex analysis.)

For any vector space W of dimension $n+1$, $\mathrm{Sym}^k W^*$ is the space of homogeneous polynomials of degree k on the projective space $\mathbb{P}^n = \mathbb{P}W$ of lines in W; dually, $\mathrm{Sym}^k W$ will be the space of homogeneous polynomials of degree k on the projective space $\mathbb{P}^n = \mathbb{P}(W^*)$ of lines in W^*, or of hyperplanes in W. Thus, the projective space $\mathbb{P}(\mathrm{Sym}^k W)$ is the space of hypersurfaces of degree k in $\mathbb{P}^n = \mathbb{P}(W^*)$. (Because of this duality, we usually work with objects in the projective space $\mathbb{P}(W^*)$ rather than the dual space $\mathbb{P}W$ in order to derive results about symmetric powers $\mathrm{Sym}^k W$; this may seem initially more confusing, but we believe it is ultimately less so.)

For any vector space V and any positive integer n, we have a natural map, called the *Veronese embedding*

$$\mathbb{P}V^* \hookrightarrow \mathbb{P}(\mathrm{Sym}^n V^*)$$

that maps the line spanned by $v \in V^*$ to the line spanned by $v^n \in \mathrm{Sym}^n V^*$. We will encounter the Veronese embedding of higher-dimensional vector spaces in later lectures; here we are concerned just with the case where V is two dimensional, so $\mathbb{P}V^* = \mathbb{P}^1$. In this case we have a map

$$\iota_n\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^n = \mathbb{P}(\mathrm{Sym}^n V^*)$$

whose image is called the *rational normal curve* $C = C_n$ *of degree* n. Choosing bases $\{\alpha, \beta\}$ for V^* and $\{\ldots[n!/k!(n-k)!]\alpha^k\beta^{n-k}\ldots\}$ for $\mathrm{Sym}^n V^*$ and expanding out $(x\alpha + y\beta)^n$ we see that in coordinates this map may be given as

$$[x, y] \mapsto [x^n, x^{n-1}y, x^{n-2}y^2, \ldots, xy^{n-1}, y^n].$$

From the definition, the action of $\mathrm{PGL}_2\mathbb{C}$ on $\mathbb{P}^n$ preserves C_n; conversely, since any automorphism of $\mathbb{P}^n$ fixing C_n pointwise is the identity, from Fact 11.15 it follows that *the group G of automorphisms of $\mathbb{P}^n$ that preserve C_n is precisely* $\mathrm{PGL}_2\mathbb{C}$. (Note that conversely if W is any $(n+1)$-dimensional representation of $\mathrm{SL}_2\mathbb{C}$ and $\mathbb{P}W \cong \mathbb{P}^n$ contains a rational normal curve of degree n preserved by the action of $\mathrm{PGL}_2\mathbb{C}$, then we must have $W \cong \mathrm{Sym}^n V$; we leave this as an exercise.[2])

When $n = 2$, C is the *plane conic* defined by the equation

[2] Note that any confusion between $\mathbb{P}W$ and $\mathbb{P}W^*$ is relatively harmless for us here, since the representations $\mathrm{Sym}^n V$ are isomorphic to their duals.

$$F(Z_0, Z_1, Z_2) = Z_0Z_2 - Z_1^2 = 0.$$

For $n = 3$, C is the *twisted cubic curve* in $\mathbb{P}^3$, and is defined by three quadratic polynomials

$$Z_0Z_2 - Z_1^2, \qquad Z_0Z_3 - Z_1Z_2, \qquad \text{and} \qquad Z_1Z_3 - Z_2^2.$$

More generally, the rational normal curve is the common zero locus of the 2×2 minors of the matrix

$$M = \begin{pmatrix} Z_0 Z_1 \dots Z_{n-1} \\ Z_1 Z_2 \dots \ Z_n \end{pmatrix},$$

that is, the locus where the rank of M is 1.

Back to Plethysm

We start with Example (11.12). We can interpret the decomposition given there (or rather the decomposition of the representation of the corresponding Lie group $\mathrm{SL}_2\mathbb{C}$) geometrically via the Veronese embedding $\iota_2\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$. As noted, $\mathrm{SL}_2\mathbb{C}$ acts on $\mathbb{P}^2 = \mathbb{P}(\mathrm{Sym}^2 V^*)$ as the group of motions of $\mathbb{P}^2$ carrying the conic curve C_2 into itself. Its action on the space $\mathrm{Sym}^2(\mathrm{Sym}^2 V))$ of quadratic polynomials on $\mathbb{P}^2$ thus must preserve the one-dimensional subspace $\mathbb{C} \cdot F$ spanned by the polynomial F above that defines the conic C_2. At the same time, we see that pullback via ι_2 defines a map from the space of quadratic polynomials on $\mathbb{P}^2$ to the space of quartic polynomials on $\mathbb{P}^1$, which has kernel $\mathbb{C} \cdot F$; thus, we have an exact sequence

$$0 \to \mathbb{C} = \mathrm{Sym}^0 V \to \mathrm{Sym}^2(\mathrm{Sym}^2 V)) \to \mathrm{Sym}^4 V \to 0,$$

which implies the decomposition of $\mathrm{Sym}^2(\mathrm{Sym}^2 V))$ described above.

Note that what comes to us at first glance is not actually the direct sum decomposition (11.12) of $\mathrm{Sym}^2(\mathrm{Sym}^2 V))$, but just the exact sequence above. The splitting of this sequence of $\mathrm{SL}_2\mathbb{C}$-modules, guaranteed by the general theory, is less obvious. For example, we are saying that given a conic curve C in the plane $\mathbb{P}^2$, there is a subspace U_C of the space of all conics in $\mathbb{P}^2$, complementary to the one-dimensional subspace spanned by C itself and invariant under the action of the group of motions of the plane $\mathbb{P}^2$ carrying C into itself. Is there a geometric description of this space? Yes: the following proposition gives one.

Proposition 11.16. *The subrepresentation* $\mathrm{Sym}^4 V \subset \mathrm{Sym}^2(\mathrm{Sym}^2 V)$ *is the space of conics spanned by the family of double lines tangent to the conic* $C = C_2$.

Proof. One way to prove this is to simply write out this subspace in coordinates: in terms of homogeneous coordinates Z_i on $\mathbb{P}^2$ as above, the tangent line to the conic C at the point $[1, \alpha, \alpha^2]$ is the line

$$L_\alpha = \{Z : \alpha^2 Z_0 - 2\alpha Z_1 + Z_2 = 0\}.$$

The double line $2L_\alpha$ is, thus, the conic with equation

$$\alpha^4 Z_0^2 - 4\alpha^3 Z_0 Z_1 + 2\alpha^2 Z_0 Z_2 + 4\alpha^2 Z_1^2 - 4\alpha Z_1 Z_2 + Z_2^2 = 0.$$

The subspace these conics generate is thus spanned by Z_0^2, Z_0Z_1, Z_1Z_2, Z_2^2, and $Z_0Z_2 + 2Z_1^2$. By construction, this is invariant under the action of $\mathrm{SL}_2\mathbb{C}$, and it is visibly complementary to the trivial subrepresentation $\mathbb{C}\cdot F = \mathbb{C}\cdot(Z_0Z_2 - Z_1^2)$.

For those familiar with some algebraic geometry, it may not be necessary to write all this down in coordinates: we could just observe that the map from the conic curve C to the projective space $\mathbb{P}(\mathrm{Sym}^2(\mathrm{Sym}^2 V))$ of conics in $\mathbb{P}^2$ sending each point $p \in C$ to the square of the tangent line to C at p is the restriction to C of the quadratic Veronese map $\mathbb{P}^2 \to \mathbb{P}^5$, and so has image a quartic rational normal curve. This spans a four-dimensional projective subspace of $\mathbb{P}(\mathrm{Sym}^2(\mathrm{Sym}^2 V))$, which must correspond to a subrepresentation isomorphic to $\mathrm{Sym}^4 V$. □

We will return to this notion in Exercise 11.26 below.

We can, in a similar way, describe the decomposition of all the symmetric powers of the representation $W = \mathrm{Sym}^2 V$; in the general setting, the geometric interpretation becomes quite handy. For example, we have seen that the third symmetric power decomposes

$$\mathrm{Sym}^3(\mathrm{Sym}^2 V) = \mathrm{Sym}^6 V \oplus \mathrm{Sym}^2 V.$$

This is immediate from the geometric description: the space of cubics in the plane $\mathbb{P}^2$ naturally decomposes into the space of cubics vanishing on the conic $C = C_2$, plus a complementary space isomorphic (via the pullback map ι_2^*) to the space of sextic polynomials on $\mathbb{P}^1$; moreover, since a cubic vanishing on C_2 factors into the quadratic polynomial F and a linear factor, the space of cubics vanishing on the conic curve $C \subset \mathbb{P}^2$ may be identified with the space of lines in $\mathbb{P}^2$.

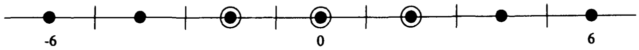
One more special case: from the general formula (11.14), we have

$$\mathrm{Sym}^4(\mathrm{Sym}^2 V) \cong \mathrm{Sym}^8 V \oplus \mathrm{Sym}^4 V \oplus \mathrm{Sym}^0 V.$$

Again, this is easy to see from the geometric picture: the space of quartic polynomials on $\mathbb{P}^2$ consists of the one-dimensional space of quartics spanned by the square of the defining equation F of C itself, plus the space of quartics vanishing on C modulo multiples of F^2, plus the space of quartics modulo those vanishing on C. (We use the word "plus," suggesting a direct sum, but as before only an exact sequence is apparent).

Exercise 11.17. Show that, in general, the order of vanishing on C defines a filtration on the space of polynomials of degree n in $\mathbb{P}^2$, whose successive quotients are the direct sum factors on the right hand side of the decomposition of Exercise 11.14.

We can similarly analyze symmetric powers of the representation $U = \mathrm{Sym}^3 V$. For example, since U has eigenvalues $-3, -1, 1$, and 3, the symmetric square of U has eigenvalues $-6, -4, -2$ (twice), 0 (twice), 2 (twice), 4, and 6; diagrammatically, we have



This implies that

$$\mathrm{Sym}^2(\mathrm{Sym}^3 V) \cong \mathrm{Sym}^6 V \oplus \mathrm{Sym}^2 V. \tag{11.18}$$

We can interpret this in terms of the twisted cubic $C = C_3 \subset \mathbb{P}^3$ as follows: the space of quadratic polynomials on $\mathbb{P}^3$ contains, as a subrepresentation, the three-dimensional vector space of quadrics containing the curve C itself; and the quotient is isomorphic, via the pullback map ι_3^*, to the space of sextic polynomials on $\mathbb{P}^1$.

Exercise 11.19*. By the above, the action of $\mathrm{SL}_2\mathbb{C}$ on the space of quadric surfaces containing the twisted cubic curve C is the same as its action on $\mathbb{P}(\mathrm{Sym}^2 V^*) \cong \mathbb{P}^2$. Make this explicit by associating to every quadric containing C a polynomial of degree 2 on $\mathbb{P}^1$, up to scalars.

Exercise 11.20*. The direct sum decomposition (11.18) says that there is a linear space of quadric surfaces in $\mathbb{P}^3$ preserved under the action of $\mathrm{SL}_2\mathbb{C}$ and complementary to the space of quadrics containing C. Describe this space.

Exercise 11.21. The projection map from $\mathrm{Sym}^2(\mathrm{Sym}^3 V)$ to $\mathrm{Sym}^2 V$ given by the decomposition (11.18) above may be viewed as a *quadratic* map from the vector space $\mathrm{Sym}^3 V$ to the vector space $\mathrm{Sym}^2 V$. Show that it may be given in these terms as the *Hessian*, that is, by associating to a homogeneous cubic polynomial in two variables the determinant of the 2×2 matrix of its second partials.

Exercise 11.22. The map in the preceding exercise may be viewed as associating to an unordered triple of points $\{p, q, r\}$ in $\mathbb{P}^1$ an unordered pair of points $\{s, t\} \subset \mathbb{P}^1$. Show that this pair of points is the pair of fixed points of the automorphism of $\mathbb{P}^1$ permuting the three points p, q, and r cyclically.

Exercise 11.23*. Show that

$$\mathrm{Sym}^3(\mathrm{Sym}^3 V) = \mathrm{Sym}^9 V \oplus \mathrm{Sym}^5 V \oplus \mathrm{Sym}^3 V,$$

and interpret this in terms of the geometry of the twisted cubic curve. In particular, show that the space of cubic surfaces containing the curve is the direct sum of the last two factors, and identify the subspace of cubics corresponding to the last factor.

Exercise 11.24. Analyze the representation $\mathrm{Sym}^4(\mathrm{Sym}^3 V)$ similarly. In particular, show that it contains a trivial one-dimensional subrepresentation.

The trivial subrepresentation of $\mathrm{Sym}^4(\mathrm{Sym}^3 V)$ found in the last exercise has an interesting interpretation. To say that $\mathrm{Sym}^4(\mathrm{Sym}^3 V)$ has such an invariant one-dimensional subspace is to say that *there exists a quartic surface in* $\mathbb{P}^3$ *preserved under all motions of* $\mathbb{P}^3$ *carrying the rational normal curve* $C = C_3$ *into itself.* What is this surface? The answer is simple: it is the *tangent developable* to the twisted cubic, that is, the surface given as the union of the tangent lines to C.

Exercise 11.25*. Show that the representation $\mathrm{Sym}^3(\mathrm{Sym}^4 V)$ contains a trivial subrepresentation, and interpret this geometrically.

Problem 11.26. Another way of interpreting the direct sum decomposition of $\mathrm{Sym}^2(\mathrm{Sym}^2 V)$ geometrically is to say that given a conic curve $C \subset \mathbb{P}^2$ and given four points on C, we can find a conic $C' = C'(C; p_1, \ldots, p_4) \subset \mathbb{P}^2$ intersecting C in exactly these points, in a way that is preserved by the action of the group $\mathrm{PGL}_3\mathbb{C}$ of all motions of $\mathbb{P}^2$ (i.e., for any motion $A\colon \mathbb{P}^2 \to \mathbb{P}^2$ of the plane, we have $A(C'(C; p_1, \ldots, p_4)) = C'(AC; Ap_1, \ldots, Ap_4))$. What is a description of this process? In particular, show that the cross-ratio of the four points p_i on the curve C' must be a function of the cross-ratio of the p_i on C, and find this function. Observe also that this process gives an endomorphism of the pencil

$$\{C \subset \mathbb{P}^2 \colon p_1, \ldots, p_4 \in C\} \cong \mathbb{P}^1$$

of conics passing through any four points $p_i \in \mathbb{P}^2$. What is the degree of this endomorphism?

The above questions have all dealt with the symmetric powers of $\mathrm{Sym}^n V$. There are also interesting questions about the exterior powers of $\mathrm{Sym}^n V$. To start with, consider the exterior square $\wedge^2(\mathrm{Sym}^3 V)$. The eigenvalues of this representation are just the pairwise sums of distinct elements of $\{3, 1, -1, -3\}$, that is, 4, 2, 0 (twice), -2, and -4; we deduce that

$$\wedge^2(\mathrm{Sym}^3 V) \cong \mathrm{Sym}^4 V \oplus \mathrm{Sym}^0 V. \tag{11.27}$$

Observe in particular that according to this there is a skew-symmetric bilinear form on the space $U = \mathrm{Sym}^3 V$ preserved (up to scalars) by the action of $\mathrm{SL}_2\mathbb{C}$. What is this form? One way of describing it would be in terms of the twisted cubic: the map from C to the dual projective space $(\mathbb{P}^3)^*$ sending each point $p \in C$ to the osculating plane to C at p extends to a skew-symmetric linear isomorphism of $\mathbb{P}^3$ with $(\mathbb{P}^3)^*$.

Exercise 11.28. Show that a line in $\mathbb{P}^3$ is isotropic for this form if and only if, viewed as an element of $\mathbb{P}(\wedge^2 U)$, it lies in the linear span of the locus of tangent lines to the twisted cubic.

Exercise 11.29. Show that the projection on the first factor in the decomposition (11.27) is given explicitly by the map

$$F \wedge G \mapsto F \cdot dG - G \cdot dF$$

and say precisely what this means.

Exercise 11.30. Show that, in general, the representation $\bigwedge^2(\mathrm{Sym}^n V)$ has as a direct sum factor the representation $\mathrm{Sym}^{2n-2} V$, and that the projection on this factor is given as in the preceding exercise. Find the remaining factors of $\bigwedge^2(\mathrm{Sym}^n V)$, and interpret them.

More on Rational Normal Curves

Exercise 11.31. Analyze in general the representations $\mathrm{Sym}^2(\mathrm{Sym}^n V)$; show, using eigenvalues, that we have

$$\mathrm{Sym}^2(\mathrm{Sym}^n V) = \bigoplus_{\alpha \geq 0} \mathrm{Sym}^{2n-4\alpha} V.$$

Exercise 11.32*. Interpret the space $\mathrm{Sym}^2(\mathrm{Sym}^n V)$ of the preceding exercise as the space of quadrics in the projective space $\mathbb{P}^n$, and use the geometry of the rational normal curve $C = C_n \subset \mathbb{P}^n$ to interpret the decomposition of this representation into irreducible factors. In particular, show that direct sum

$$\bigoplus_{\alpha \geq 1} \mathrm{Sym}^{2n-4\alpha} V$$

is the space of quadratic polynomials vanishing on the rational normal curve; and that the direct sum

$$\bigoplus_{\alpha \geq 2} \mathrm{Sym}^{2n-4\alpha} V$$

is the space of quadrics containing the *tangential developable* of the rational normal curve, that is, the union of the tangent lines to C. Can you interpret the sums for $\alpha \geq k$ for $k > 2$?

Exercise 11.33. Note that by Exercise 11.11, the tensor power

$$\mathrm{Sym}^n V \otimes \mathrm{Sym}^n V$$

always contains a copy of the trivial representation; and that by Exercises 11.30 and 11.31, this trivial subrepresentation will lie in $\mathrm{Sym}^2(\mathrm{Sym}^n V)$ if n is even and in $\bigwedge^2(\mathrm{Sym}^n V)$ if n is odd. Show that in either case, the bilinear form on $\mathrm{Sym}^n V$ preserved by $\mathrm{SL}_2\mathbb{C}$ may be described as the isomorphism of $\mathbb{P}^n$ with $(\mathbb{P}^n)^*$ carrying each point p of the rational normal curve $C \subset \mathbb{P}^n$ into the osculating hyperplane to C at p.

Comparing Exercises 11.14 and 11.31, we see that $\mathrm{Sym}^2(\mathrm{Sym}^n V) \cong \mathrm{Sym}^n(\mathrm{Sym}^2 V)$; apparently coincidentally. This is in fact a special case of a more general theorem (cf. Exercise 6.18):

Exercise 11.34. (Hermite Reciprocity). Use the eigenvalues of H to prove the isomorphism

$$\mathrm{Sym}^k(\mathrm{Sym}^n V) \cong \mathrm{Sym}^n(\mathrm{Sym}^k V).$$

Can you exhibit explicitly a map between these two?

Note that in the examples of Hermite reciprocity we have seen, it seems completely coincidental: for example, the fact that the representations $\mathrm{Sym}^3(\mathrm{Sym}^4 V)$ and $\mathrm{Sym}^4(\mathrm{Sym}^3 V)$ both contain a trivial representation corresponds to the facts that the tangential developable of the twisted cubic in $\mathbb{P}^3$ has degree 4, while the chordal variety of the rational normal quartic in $\mathbb{P}^4$ has degree 3.

Exercise 11.35*. Show that $\bigwedge^m(\mathrm{Sym}^n V) \cong \mathrm{Sym}^m(\mathrm{Sym}^{n+1-m} V)$.

We will see in Lecture 23 that there is a unique closed orbit in $\mathbb{P}(W)$ for any irreducible representation W. For now, we can do the following special case.

Exercise 11.36. Show that the unique closed orbit of the action of $\mathrm{SL}_2\mathbb{C}$ on the projectivization of any irreducible representation is isomorphic to $\mathbb{P}^1$ (these are the *rational normal curves* introduced above).

LECTURE 12

Representations of $\mathfrak{sl}_3\mathbb{C}$, Part I

This lecture develops results for $\mathfrak{sl}_3\mathbb{C}$ analogous to those of §11.1 (though not in exactly the same order). This involves generalizing some of the basic terms of §11 (e.g., the notions of eigenvalue and eigenvector have to be redefined), but the basic ideas are in some sense already in §11. Certainly no techniques are involved beyond those of §11.1.

We come now to a second important stage in the development of the theory: in the following, we will take our analysis of the representations of $\mathfrak{sl}_2\mathbb{C}$ and see how it goes over in the next case, the algebra $\mathfrak{sl}_3\mathbb{C}$. As we will see, a number of the basic constructions need to be modified, or at least rethought. There are, however, two pieces of good news that should be borne in mind. First, we will arrive, by the end of the following lecture, at a classification of the representations of $\mathfrak{sl}_3\mathbb{C}$ that is every bit as detailed and explicit as the classification we arrived at previously for $\mathfrak{sl}_2\mathbb{C}$. Second, once we have redone our analysis in this context, *we will need to introduce no further concepts to carry out the classification of the finite-dimensional representations of all remaining semisimple Lie algebras.*

We will proceed by analogy with the previous lecture. To begin with, we started out our analysis of $\mathfrak{sl}_2\mathbb{C}$ with the basis $\{H, X, Y\}$ for the Lie algebra; we then proceeded to decompose an arbitrary representation V of $\mathfrak{sl}_2\mathbb{C}$ into a direct sum of eigenspaces for the action of H. What element of $\mathfrak{sl}_3\mathbb{C}$ in particular will play the role of H? The answer—and this is the first and perhaps most wrenching change from the previous case—is that no one element really allows us to see what is going on.[1] Instead, we have to replace

[1] This is not literally true: as we will see from the following analysis, if H is any diagonal matrix whose entries are independent over $\mathbb{Q}$, then the action of H on any representation V of $\mathfrak{sl}_3\mathbb{C}$ determines the representation (i.e., if we know the eigenvalues of H we know V). But (as we will also see) trying to carry this out in practice would be sheer perversity.

the single element $H \in \mathfrak{sl}_2\mathbb{C}$ with a *subspace* $\mathfrak{h} \subset \mathfrak{sl}_3\mathbb{C}$, namely, the two-dimensional subspace of all diagonal matrices. The idea is a basic one; it comes down to the observation that *commuting diagonalizable matrices are simultaneously diagonalizable.* This translates in the present circumstances to the statement that any finite-dimensional representation V of $\mathfrak{sl}_3\mathbb{C}$ admits a decomposition $V = \bigoplus V_\alpha$, where every vector $v \in V_\alpha$ is an eigenvector for every element $H \in \mathfrak{h}$.

At this point some terminology is clearly in order, since we will be dealing with the action not of a single matrix H but rather a vector space $\mathfrak{h}$ of them. To begin with, by an *eigenvector* for $\mathfrak{h}$ we will mean, reasonably enough, a vector $v \in V$ that is an eigenvector for every $H \in \mathfrak{h}$. For such a vector v we can write

$$H(v) = \alpha(H)\cdot v, \tag{12.1}$$

where $\alpha(H)$ is a scalar depending linearly on H, i.e., $\alpha \in \mathfrak{h}^*$. This leads to our second notion: by an *eigenvalue* for the action of $\mathfrak{h}$ we will mean an element $\alpha \in \mathfrak{h}^*$ such that there exists a nonzero element $v \in V$ satisfying (12.1); and by the *eigenspace* associated to the eigenvalue α we will mean the subspace of all vectors $v \in V$ satisfying (12.1). Thus we may phrase the statement above as

(12.2) *Any finite-dimensional representation V of $\mathfrak{sl}_3\mathbb{C}$ has a decomposition*

$$V = \bigoplus V_\alpha,$$

where V_α is an eigenspace for $\mathfrak{h}$ and α ranges over a finite subset of $\mathfrak{h}^*$.

This is, in fact, a special case of a more general statement: for any semisimple Lie algebra $\mathfrak{g}$, we will be able to find an abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$, such that the action of $\mathfrak{h}$ on any $\mathfrak{g}$-module V will be diagonalizable, i.e., we will have a direct sum decomposition of V into eigenspaces V_α for $\mathfrak{h}$.

Having decided what the analogue for $\mathfrak{sl}_3\mathbb{C}$ of $H \in \mathfrak{sl}_2\mathbb{C}$ is, let us now consider what will play the role of X and Y. The key here is to look at the commutation relations

$$[H, X] = 2X \quad \text{and} \quad [H, Y] = -2Y$$

in $\mathfrak{sl}_2\mathbb{C}$. The correct way to interpret these is as saying that *X and Y are eigenvectors for the adjoint action of H on* $\mathfrak{sl}_2\mathbb{C}$. In our present circumstances, then, we want to look for eigenvectors (in the new sense) for the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}_3\mathbb{C}$. In other words, we apply (12.2) to the adjoint representation of $\mathfrak{sl}_3\mathbb{C}$ to obtain a decomposition

$$\mathfrak{sl}_3\mathbb{C} = \mathfrak{h} \oplus \left(\bigoplus \mathfrak{g}_\alpha\right), \tag{12.3}$$

where α ranges over a finite subset of $\mathfrak{h}^*$ and $\mathfrak{h}$ acts on each space $\mathfrak{g}_\alpha$ by scalar multiplication, i.e., for any $H \in \mathfrak{h}$ and $Y \in \mathfrak{g}_\alpha$,

$$[H, Y] = \operatorname{ad}(H)(Y) = \alpha(H)\cdot Y.$$

This is probably easier to carry out in practice than it is to say; we are being longwinded here because once this process is understood it will be straightforward to apply it to the other Lie algebras. In any case, to do it in the present circumstances, we just observe that multiplication of a matrix M on the left by a diagonal matrix D with entries a_i multiplies the ith row of M by a_i, while multiplication on the right multiplies the ith column by a_i; if the entries of M are $m_{i,j}$, the entries of the commutator $[D, M]$ are thus $(a_i - a_j)m_{i,j}$. We see then that the commutator $[D, M]$ will be a multiple of M for all D if and only if all but one entry of M are zero. Thus, if we let $E_{i,j}$ be the 3×3 matrix whose (i, j)th entry is 1 and all of whose other entries are 0, we see that the $E_{i,j}$ exactly generate the eigenspaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$.

Explicitly, we have

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} : a_1 + a_2 + a_3 = 0 \right\}$$

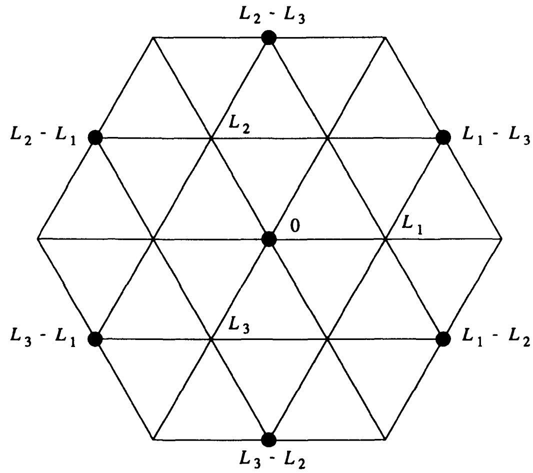
and so we can write

$$\mathfrak{h}^* = \mathbb{C}\{L_1, L_2, L_3\}/(L_1 + L_2 + L_3 = 0)\},$$

where

$$L_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i.$$

The linear functionals $\alpha \in \mathfrak{h}^*$ appearing in the direct sum decomposition (12.3) are thus the six functionals $L_i - L_j$; the space $\mathfrak{g}_{L_i - L_j}$ will be generated by the element $E_{i,j}$. To draw a picture



(12.4)

The virtue of this decomposition and the corresponding picture is that we can read off from it pretty much the entire structure of the Lie algebra. Of course, the action of $\mathfrak{h}$ on $\mathfrak{g}$ is clear from the picture: $\mathfrak{h}$ carries each of the subspaces $\mathfrak{g}_\alpha$ into itself, acting on each $\mathfrak{g}_\alpha$ by scalar multiplication by the linear functional represented by the corresponding dot. Beyond that, though, we can also see, much as in the case of representations of $\mathfrak{sl}_2\mathbb{C}$, how the rest of the Lie algebra acts. Basically, we let X be any element of $\mathfrak{g}_\alpha$ and ask where $\mathrm{ad}(X)$ sends a given vector $Y \in \mathfrak{g}_\beta$; the answer as before comes from knowing how $\mathfrak{h}$ acts on $\mathrm{ad}(X)(Y)$. Explicitly, we let H be an arbitrary element of $\mathfrak{h}$ and as on page 148 we make the

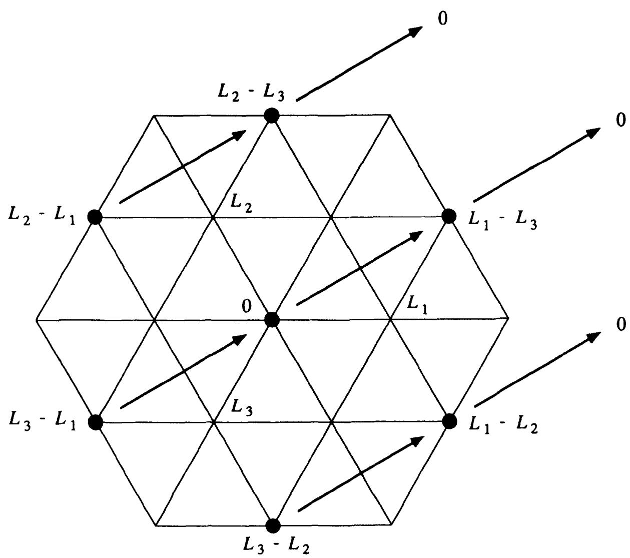
Fundamental Calculation (*second time*):

$$\begin{aligned}[H,[X,Y]] &= [X,[H,Y]] + [[H,X],Y] \\ &= [X, \beta(H)\cdot Y] + [\alpha(H)\cdot X, Y] \\ &= (\alpha(H) + \beta(H))\cdot[X,Y].\end{aligned}$$

In other words, $[X,Y] = \mathrm{ad}(X)(Y)$ *is again an eigenvector for* $\mathfrak{h}$, *with eigenvalue* $\alpha + \beta$. Thus,

$$\mathrm{ad}(\mathfrak{g}_\alpha)\colon \mathfrak{g}_\beta \to \mathfrak{g}_{\alpha+\beta};$$

in particular, the action of $\mathrm{ad}(\mathfrak{g}_\alpha)$ preserves the decomposition (12.3) in the sense that it carries each eigenspace $\mathfrak{g}_\beta$ into another. We can interpret this in terms of the diagram (12.4) of eigenspaces by saying that each $\mathfrak{g}_\alpha$ acts, so to speak, by "translation"; that is, it carries each space $\mathfrak{g}_\beta$ corresponding to a dot in the diagram into the subspace $\mathfrak{g}_{\alpha+\beta}$ corresponding to that dot translated by α. For example, the action of $\mathfrak{g}_{L_1-L_3}$ may be pictured as



(12.5)

i.e., it carries $\mathfrak{g}_{L_2-L_1}$ into $\mathfrak{g}_{L_2-L_3}$; $\mathfrak{g}_{L_3-L_1}$ into $\mathfrak{h}$; $\mathfrak{h}$ into $\mathfrak{g}_{L_1-L_3}$, $\mathfrak{g}_{L_3-L_2}$ into $\mathfrak{g}_{L_1-L_2}$, and kills $\mathfrak{g}_{L_2-L_3}$, $\mathfrak{g}_{L_1-L_3}$, and $\mathfrak{g}_{L_1-L_2}$. Of course, not all the data can be read off of the diagram, at least on the basis on what we have said so far. For example, we do not at present see from the diagram the kernel of $\mathrm{ad}(\mathfrak{g}_{L_1-L_3})$ on $\mathfrak{h}$, though we will see later how to read this off as well. We do, however, have at least a pretty good idea of who is doing what to whom.

Pretty much the same picture applies to any representation V of $\mathfrak{sl}_3\mathbb{C}$: we start from the eigenspace decomposition $V = \bigoplus V_\alpha$ for the action of $\mathfrak{h}$ that we saw in (12.2). Next, the commutation relations for $\mathfrak{sl}_3\mathbb{C}$ tell us exactly how the remaining summands of the decomposition (12.3) of $\mathfrak{sl}_3\mathbb{C}$ act on the space V, and again we will see that each of the spaces $\mathfrak{g}_\alpha$ acts by carrying one eigenspace V_β into another. As usual, for any $X \in \mathfrak{g}_\alpha$ and $v \in V_\beta$ we can tell where X will send v if we know how an arbitrary element $H \in \mathfrak{h}$ will act on $X(v)$. This we can determine by making the

Fundamental Calculation (*third time*):

$$\begin{aligned} H(X(v)) &= X(H(v)) + [H, X](v) \\ &= X(\beta(H)\cdot v) + (\alpha(H)\cdot X)(v) \\ &= (\alpha(H) + \beta(H))\cdot X(v). \end{aligned}$$

We see from this that $X(v)$ *is again an eigenvector for the action of* $\mathfrak{h}$, *with eigenvalue* $\alpha + \beta$; in other words, the action of $\mathfrak{g}_\alpha$ carries V_β to $V_{\alpha+\beta}$. We can thus represent the eigenspaces V_α of V by dots in a plane diagram so that each $\mathfrak{g}_\alpha$ acts again "by translation," as we did for representations of $\mathfrak{sl}_2\mathbb{C}$ in the preceding lecture and the adjoint representation of $\mathfrak{sl}_3\mathbb{C}$ above. Just as in the case of $\mathfrak{sl}_2\mathbb{C}$ (page 148), we have

Observation 12.6. *The eigenvalues α occurring in an irreducible representation of $\mathfrak{sl}_3\mathbb{C}$ differ from one other by integral linear combinations of the vectors $L_i - L_j \in \mathfrak{h}^*$.*

Note that these vectors $L_i - L_j$ generate a lattice in $\mathfrak{h}^*$, which we will denote by Λ_R, and that all the α lie in some translate of this lattice.

At this point, we should begin to introduce some of the terminology that appears in this subject. The basic object here, the eigenvalue $\alpha \in \mathfrak{h}^*$ of the action of $\mathfrak{h}$ on a representation V of $\mathfrak{g}$, is called a *weight* of the representation; the corresponding eigenvectors in V_α are called, naturally enough, *weight vectors* and the spaces V_α themselves *weight spaces*. Clearly, the weights that occur in the adjoint representation are special; these are called the *roots* of the Lie algebra and the corresponding subspaces $\mathfrak{g}_\alpha \subset \mathfrak{g}$ *root spaces*; by convention, zero is not a root. The lattice $\Lambda_R \subset \mathfrak{h}^*$ generated by the roots α is called the *root lattice.*

To see what the next step should be, we go back to the analysis of representations of $\mathfrak{sl}_2\mathbb{C}$. There, at this stage we continued our analysis by going to an extremal eigenspace V_α and taking a vector $v \in V_\alpha$. The point was that since V_α was extremal, the operator X, which would carry V_α to $V_{\alpha+2}$, would have to kill v; so that v would be then both an eigenvector for H and in the kernel of X. We then saw that these two facts allowed us to completely describe the representation V in terms of images of v.

What would be the appropriately analogous setup in the case of $\mathfrak{sl}_3\mathbb{C}$? To start at the beginning, there is the question of what we mean by *extremal*: in the case of $\mathfrak{sl}_2\mathbb{C}$, since we knew that all the eigenvalues were scalars differing by integral multiples of 2, there was not much ambiguity about what we meant by this. In the present circumstance this does involve a priori a choice (though as we shall see the choice does not affect the outcome): we have to choose a direction, and look for the farthest α in that direction appearing in the decomposition (12.3). What this means is that we should choose a linear functional

$$l\colon \Lambda_R \to \mathbb{R},$$

extend it by linearity to a linear functional $l\colon \mathfrak{h}^* \to \mathbb{C}$, and then for any representation V we should go to the eigenspace V_α for which the real part of $l(\alpha)$ is maximal.[2] Of course, to avoid ambiguity we should choose l to be irrational with respect to the lattice Λ_R, that is, to have no kernel.

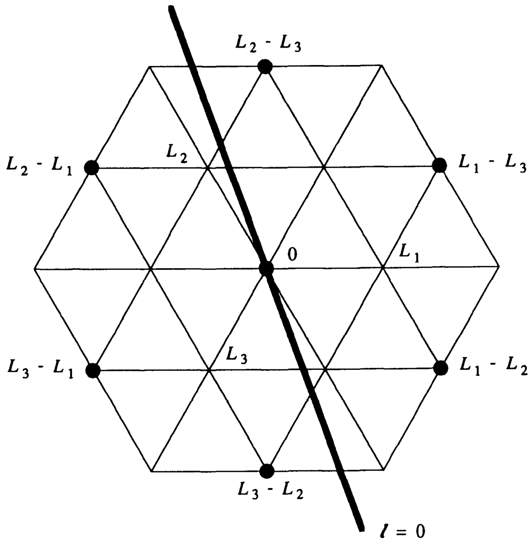
What is the point of this? The answer is that, just as in the case of a representation V of $\mathfrak{sl}_2\mathbb{C}$ we found in this way a vector $v \in V$ that was simultaneously in the kernel of the operator X and an eigenvector for H, in the present case what we will find is a vector $v \in V_\alpha$ that is an eigenvector for $\mathfrak{h}$, *and at the same time in the kernel of the action of* $\mathfrak{g}_\beta$ *for every* β *such that* $l(\beta) > 0$—that is, *that is killed by half the root spaces* $\mathfrak{g}_\beta$ (specifically, the root spaces corresponding to dots in the diagram (12.4) lying in a half plane). This will likewise give us a nearly complete description of the representation V.

To carry this out explicitly, choose our functional l to be given by

$$l(a_1L_1 + a_2L_2 + a_3L_3) = aa_1 + ba_2 + ca_3,$$

where $a + b + c = 0$ and $a > b > c$, so that the spaces $\mathfrak{g}_\alpha \subset \mathfrak{g}$ for which we have $l(\alpha) > 0$ are then exactly $\mathfrak{g}_{L_1-L_3}$, $\mathfrak{g}_{L_2-L_3}$, and $\mathfrak{g}_{L_1-L_2}$; they correspond to matrices with one nonzero entry above the diagonal.

[2] The real-versus-complex business is a red herring since (it will turn out very shortly) all the eigenvalues α actually occurring in any representation will in fact be in the real (in fact, the rational) linear span of Λ_R.



(12.7)

Thus, for $i < j$, the matrices $E_{i,j}$ *generate the positive root spaces*, and the $E_{j,i}$ *generate the negative root spaces*. We set

$$H_{i,j} = [E_{i,j}, E_{j,i}] = E_{i,i} - E_{j,j}. \tag{12.8}$$

Now let V be any irreducible, finite-dimensional representation of $\mathfrak{sl}_3\mathbb{C}$. The upshot of all the above is the

Lemma 12.9. *There is a vector $v \in V$ with the properties that*

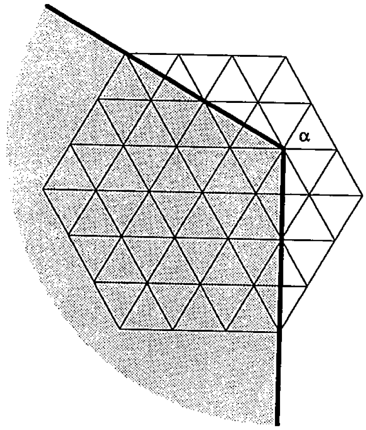
(i) *v is an eigenvector for $\mathfrak{h}$, i.e. $v \in V_\alpha$ for some α; and*
(ii) *v is killed by $E_{1,2}$, $E_{1,3}$, and $E_{2,3}$.*

For any representation V of $\mathfrak{sl}_3\mathbb{C}$, a vector $v \in V$ with these properties is called a *highest weight vector*.

In the case of $\mathfrak{sl}_2\mathbb{C}$, having found an eigenvector v for H killed by X, we argued that the images of v under successive applications of Y generated the representation. The situation here is the same: analogous to Claim 11.4 we have

Claim 12.10. *Let V be an irreducible representation of $\mathfrak{sl}_3\mathbb{C}$, and $v \in V$ a highest weight vector. Then V is generated by the images of v under successive applications of the three operators $E_{2,1}$, $E_{3,1}$, and $E_{3,2}$.*

Before we check the claim, we note three immediate consequences. First, it says that all the eigenvalues $\beta \in \mathfrak{h}^*$ occurring in V lie in a sort of $\frac{1}{3}$-plane with corner at α:



Second, we see that the dimension of V_α itself is 1, so that v is the unique eigenvector with this eigenvalue (up to scalars, of course). (We will see below that in fact v is the unique highest weight vector of V up to scalars; see Proposition 12.11.) Lastly, it says that the spaces $V_{\alpha+n(L_2-L_1)}$ and $V_{\alpha+n(L_3-L_2)}$ are all at most one dimensional, since they must be spanned by $(E_{2,1})^n(v)$ and $(E_{3,2})^n(v)$, respectively.

Proof of Claim 12.10. This is formally the same as the proof of the corresponding statement for $\mathfrak{sl}_2\mathbb{C}$: we argue that the subspace W of V spanned by images of v under the subalgebra of $\mathfrak{sl}_3\mathbb{C}$ generated by $E_{2,1}$, $E_{3,1}$, and $E_{3,2}$ is, in fact, preserved by all of $\mathfrak{sl}_3\mathbb{C}$ and hence must be all of V. To do this we just have to check that $E_{1,2}$, $E_{2,3}$, and $E_{1,3}$ carry W into itself (in fact it is enough to do this for the first two, the third being their commutator), and this is straightforward. To begin with, v itself is in the kernel of $E_{1,2}$, $E_{2,3}$, and $E_{1,3}$, so there is no problem there. Next we check that $E_{2,1}(v)$ is kept in W: we have

$$\begin{aligned} E_{1,2}(E_{2,1}(v)) &= (E_{2,1}(E_{1,2}(v)) + [E_{1,2}, E_{2,1}](v) \\ &= \alpha([E_{1,2}, E_{2,1}])\cdot v \end{aligned}$$

since $E_{1,2}(v) = 0$ and $[E_{1,2}, E_{2,1}] \in \mathfrak{h}$; and

$$\begin{aligned} E_{2,3}(E_{2,1}(v)) &= (E_{2,1}(E_{2,3}(v)) + [E_{2,3}, E_{2,1}](v) \\ &= 0 \end{aligned}$$

since $E_{2,3}(v) = 0$ and $[E_{2,3}, E_{2,1}] = 0$. A similar computation shows that $E_{3,2}(v)$ is also carried into V by $E_{1,2}$ and $E_{2,3}$.

More generally, we may argue the claim by a sort of induction: we let w_n denote any word of length n or less in the letters $E_{2,1}$ and $E_{3,2}$ and take W_n to be the vector space spanned by the vectors $w_n(v)$ for all such words; note that W is the union of the spaces W_n, since $E_{3,1}$ is the commutator of $E_{3,2}$ and $E_{2,1}$. We claim that $E_{1,2}$ and $E_{2,3}$ carry W_n into W_{n-1}. To see this, we can write w_n as either $E_{2,1} \circ w_{n-1}$ or $E_{3,2} \circ w_{n-1}$; in either case $w_{n-1}(v)$ will be an eigenvector for $\mathfrak{h}$ with eigenvalue β for some β. In the former case we have

$$\begin{aligned} E_{1,2}(w_n(v)) &= E_{1,2}(E_{2,1}(w_{n-1}(v))) \\ &= E_{2,1}(E_{1,2}(w_{n-1}(v))) + [E_{1,2}, E_{2,1}](w_{n-1}(v)) \\ &\in E_{2,1}(W_{n-2}) + \beta([E_{1,2}, E_{2,1}])\cdot w_{n-1}(v) \\ &\subset W_{n-1} \end{aligned}$$

since $[E_{1,2}, E_{2,1}] \in \mathfrak{h}$; and

$$\begin{aligned} E_{2,3}(w_n(v)) &= E_{2,3}(E_{2,1}(w_{n-1}(v))) \\ &= E_{2,1}(E_{2,3}(w_{n-1}(v))) + [E_{2,3}, E_{2,1}](w_{n-1}(v)) \\ &\in E_{2,1}(W_{n-2}) \\ &\subset W_{n-1} \end{aligned}$$

since $[E_{2,3}, E_{2,1}] = 0$. Essentially the same calculation covers the latter case $w_n = E_{3,2} \circ w_{n-1}$, establishing the claim. □

This argument shows a little more; in fact, it proves

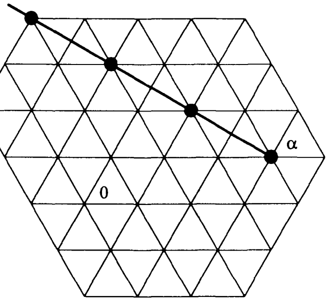
Proposition 12.11. *If V is any representation of $\mathfrak{sl}_3\mathbb{C}$ and $v \in V$ is a highest weight vector, then the subrepresentation W of V generated by the images of v by successive applications of the three operators $E_{2,1}$, $E_{3,1}$, and $E_{3,2}$ is irreducible.*

PROOF. Let α be the weight of v. The above shows that W is a subrepresentation, and it is clear that W_α is one dimensional. If W were not irreducible, we would have $W = W' \oplus W''$ for some representations W' and W''. But since projection to W' and W'' commute with the action of $\mathfrak{h}$, we have $W_\alpha = W'_\alpha \oplus W''_\alpha$. This shows that one of these spaces is zero, which implies that v belongs to W' or W'', and hence that W is W' or W''. □

As a corollary of this proposition we see that any irreducible representation of $\mathfrak{sl}_3\mathbb{C}$ has a unique highest weight vector, up to scalars; more generally, the set of highest weight vectors in V forms a union of linear subspaces Ψ_W corresponding to the irreducible subrepresentations W of V, with the dimension of Ψ_W equal to the number of times W appears in the direct sum decomposition of V into irreducibles.

What do we do next? Well, let us continue to look at the border vectors $(E_{2,1})^k(v)$. We call these border vectors because they live in (and, as we saw, span) a collection of eigenspaces $\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha+L_2-L_1}, \mathfrak{g}_{\alpha+2(L_2-L_1)}, \ldots$ that correspond to points on the boundary of the diagram above of possible eigenvalues of V. We also know that they span an uninterrupted string of nonzero eigenspaces $\mathfrak{g}_{\alpha+k(L_2-L_1)} \cong \mathbb{C}$, $k = 0, 1, \ldots$, until we get to the first m such that

$(E_{2,1})^m(v) = 0$; after that we have $\mathfrak{g}_{\alpha+k(L_2-L_1)} = (0)$ for all $k \geq m$. The picture is thus:



(12.12)

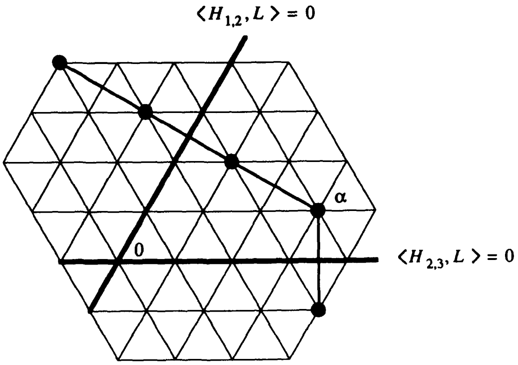
where we have no dots above/to the right of the bold line, and no dots on that line other than the ones marked.

The obvious question now is how long the string of dots along this line is. One way to answer this would be to make a calculation analogous to the one in the preceding lecture: use the computation made above to say explicitly for any k what multiple of $(E_{2,1})^{k-1}(v)$ the image of $(E_{2,1})^k(v)$ under the map $E_{1,2}$ is, and use the fact that $(E_{2,1})^m(v) = 0$ to determine m. It will be simpler—and more useful in general—if instead we just use what we have already learned about representations of $\mathfrak{sl}_2\mathbb{C}$. The point is, *the elements $E_{1,2}$ and $E_{2,1}$, together with their commutator $[E_{1,2}, E_{2,1}] = H_{1,2}$, span a subalgebra of $\mathfrak{sl}_3\mathbb{C}$ isomorphic to $\mathfrak{sl}_2\mathbb{C}$ via an isomorphism carrying $E_{1,2}$, $E_{2,1}$ and $H_{1,2}$ to the elements X, Y and H.* We will denote this subalgebra by $\mathfrak{s}_{L_1-L_2}$ (the notation may appear awkward, but this is a special case of a general construction). By the description we have already given of the action of $\mathfrak{sl}_3\mathbb{C}$ on the representation V in terms of the decomposition $V = \bigoplus V_\alpha$, we see that the subalgebra $\mathfrak{s}_{L_1-L_2}$ will shift eigenspaces V_α only in the direction of $L_2 - L_1$; in particular, the direct sum of the eigenspaces in question, namely the subspace

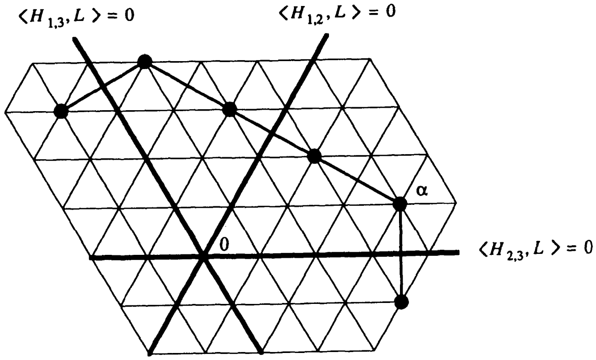
$$W = \bigoplus_k \mathfrak{g}_{\alpha+k(L_2-L_1)} \tag{12.13}$$

of V will be preserved by the action of $\mathfrak{s}_{L_1-L_2}$. In other words, W is a representation of $\mathfrak{s}_{L_1-L_2} \cong \mathfrak{sl}_2\mathbb{C}$ and we may deduce from this that *the eigenvalues of $H_{1,2}$ on W are integral, and symmetric with respect to zero.* Leaving aside the integrality for the moment, this says that the string of dots in diagram (12.12) must be symmetric with respect to the line $\langle H_{1,2}, L\rangle = 0$ in the plane $\mathfrak{h}^*$. Happily (though by no means coincidentally, as we shall see), this line is perpendicular to the line spanned by $L_1 - L_2$ in the picture we have drawn; so we can say simply that *the string of dots occurring in diagram* (12.12) *is preserved under reflection in the line* $\langle H_{1,2}, L\rangle = 0$.

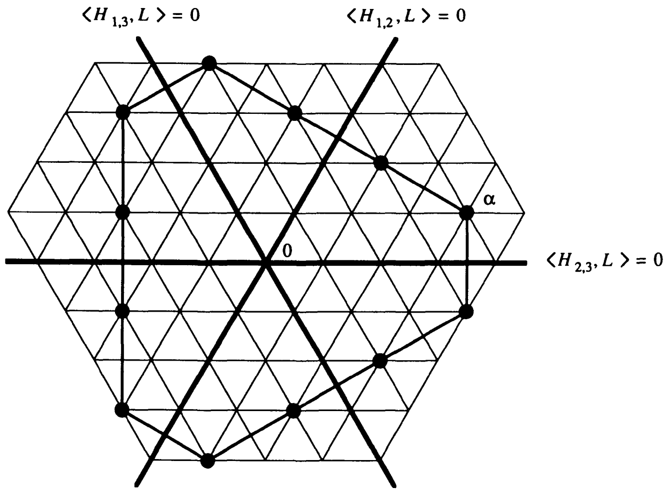
In general, for any $i \neq j$ the elements $E_{i,j}$ and $E_{j,i}$, together with their commutator $[E_{i,j}, E_{j,i}] = H_{i,j}$, span a subalgebra $\mathfrak{s}_{L_i-L_j}$ of $\mathfrak{sl}_3\mathbb{C}$ isomorphic to $\mathfrak{sl}_2\mathbb{C}$ via an isomorphism carrying $E_{i,j}$, $E_{j,i}$, and $H_{i,j}$ to the elements X, Y, and H. (Note that $H_{i,j} = -H_{j,i}$.) Analyzing the action of the subalgebra $\mathfrak{s}_{L_2-L_3}$ in particular then shows that the string of dots corresponding to the eigenspaces $\mathfrak{g}_{\alpha+k}(L_3 - L_2)$ is likewise preserved under reflection in the line $\langle H_{2,3}, L\rangle = 0$ in $\mathfrak{h}^*$. The picture is thus



Let us now take a look at the last eigenspace in the first string, that is, V_β where m is as before the smallest integer such that $(E_{2,1})^m(v) = 0$ and $\beta = \alpha + (m-1)(L_2 - L_1)$. If $v' \in V_\beta$ is any vector, then, by definition, we have $E_{2,1}(v') = 0$; and since there are no eigenspaces V_γ corresponding to γ above the bold line in diagram (12.12), we have as well that $E_{2,3}(v') = E_{1,3}(v') = 0$. Thus, v', like v itself, satisfies the statement of Lemma 12.9, except for the exchange of the indices 2 and 1; or in other words, if we had chosen the linear functional l above differently—precisely, with coefficients $b > a > c$—then the vector whose existence is implied by Lemma 12.9 would have turned out to be v' rather than v. If, indeed, we had carried out the above analysis with respect to the vector v' instead of v, we would find that all eigenvalues of V occur below or to the right of the lines through β in the directions of $L_1 - L_2$ and $L_3 - L_1$, and that the strings of eigenvalues occurring on these two lines were symmetric about the lines $\langle H_{1,2}, L\rangle = 0$ and $\langle H_{1,3}, L\rangle = 0$, respectively. The picture now is



Needless to say, we can continue to play the same game all the way around: at the end of the string of eigenvalues $\{\beta + k(L_3 - L_1)\}$ we will arrive at a vector v'' that is an eigenvector for $\mathfrak{h}$ and killed by $E_{3,1}$ and $E_{2,1}$, and to which therefore the same analysis applies. In sum, then, we see that the set of eigenvalues in V will be bounded by a hexagon symmetric with respect to the lines $\langle H_{i,j}, L\rangle = 0$ and with one vertex at α; indeed, this characterizes the hexagon as the convex hull of the union of the images of α under the group of isometries of the plane generated by reflections in these three lines.



(12.14)

We will see in a moment that the set of eigenvalues will include all the points congruent to α modulo the lattice Λ_R generated by the $L_i - L_j$ lying on the boundary of this hexagon, and that each of these eigenvalues will occur with multiplicity one.

The use of the subalgebras $\mathfrak{s}_{L_i - L_j}$ does not stop here. For one thing, observe that as an immediate consequence of our analysis of $\mathfrak{sl}_2\mathbb{C}$, all the eigenvalues of the elements $H_{i,j}$ must be integers; it is not hard to see that this means that all the eigenvalues occurring in (12.2) must be integral linear combinations of the L_i, i.e., in terms of the diagrams above, all dots must lie in the lattice Λ_W of interstices (as indeed we have been drawing them). Thus, we have

Proposition 12.15. *All the eigenvalues of any irreducible finite-dimensional representation of* $\mathfrak{sl}_3\mathbb{C}$ *must lie in the lattice* $\Lambda_W \subset \mathfrak{h}^*$ *generated by the* L_i *and be congruent modulo the lattice* $\Lambda_R \subset \mathfrak{h}^*$ *generated by the* $L_i - L_j$.

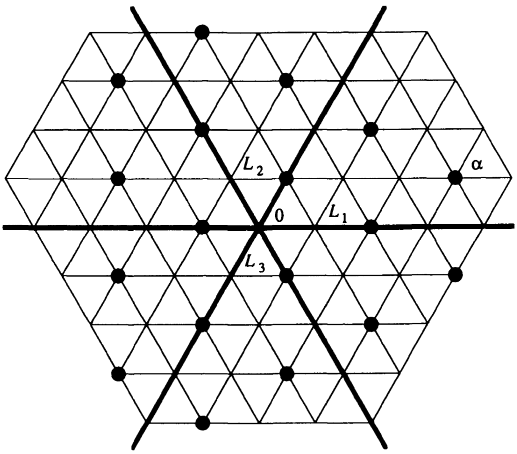
This is exactly analogous to he situation of the previous lecture: there we saw that the eigenvalues of H in any irreducible, finite-dimensional representation of $\mathfrak{sl}_2\mathbb{C}$ lay in the lattice $\Lambda_W \cong \mathbb{Z}$ of linear forms on $\mathbb{C}H$ integral on H, and were congruent to one another modulo the sublattice $\Lambda_R = 2\cdot\mathbb{Z}$ generated by the eigenvalues of H under the adjoint representation. Note that in the case of $\mathfrak{sl}_2\mathbb{C}$ we have $\Lambda_W/\Lambda_R \cong \mathbb{Z}/2$, while in the present case we have $\Lambda_W/\Lambda_R \cong \mathbb{Z}/3$; we will see later how this reflects a general pattern. The lattice Λ_W is called the *weight lattice.*

Exercise 12.16. Show that the two conditions that the eigenvalues of V are congruent to one another modulo Λ_R and are preserved under reflection in the three lines $\langle H_{i,j}, L\rangle = 0$ imply that they all lie in Λ_W, and that, in fact, this characterizes Λ_W.

To continue, we can go into the interior of the diagram (12.14) of eigenvalues of V by observing that the direct sums (12.13) are not the only visible subspaces of V preserved under the action of the subalgebras $\mathfrak{s}_{L_i-L_j}$; more generally, for any $\beta \in \mathfrak{h}^*$ appearing in the decomposition (12.2) and any i,j the direct sum

$$W = \bigoplus_k \mathfrak{g}_{\beta+k(L_i-L_j)}$$

will be a representation of $\mathfrak{s}_{L_i-L_j}$ (not necessarily irreducible, of course); in particular it follows that the values of k for which $V_{\beta+k(L_i-L_j)} \neq (0)$ form an unbroken string of integers. Observing that if β is any of the "extremal" eigenvalues pictured in diagram (12.14), then this string will include another; so that all eigenvalues congruent to the dots pictured in diagram (12.14) and lying in their convex hull must also occur. Thus, the complete diagram of eigenvalues will look like



(12.17)

We can summarize this description in

Proposition 12.18. *Let V be any irreducible, finite-dimensional representation of $\mathfrak{sl}_3\mathbb{C}$. Then for some $\alpha \in \Lambda_W \subset \mathfrak{h}^*$, the set of eigenvalues occurring in V is* *exactly the set of linear functionals congruent to α modulo the lattice Λ_R and lying in the hexagon with vertices the images of α under the group generated by reflections in the lines $\langle H_{i,j}, L\rangle = 0$.*

Remark. We did, in the analysis thus far, make one apparently arbitrary choice when we defined the notion of "extremal" eigenvalue by choosing a linear functional l on $\mathfrak{h}^*$. We remark here that, in fact, the choice was not as broad as might at first have appeared. Indeed, given the fact that the configuration of eigenvalues occurring in any irreducible finite-dimensional representation of $\mathfrak{sl}_3\mathbb{C}$ is always either a triangle or a hexagon, the "extremal" eigenvalue picked out by l will always turn out to be one of the three or six vertices of this figure; in other words, if we define the linear functional l to take $a_1L_1 + a_2L_2 + a_3L_3$ to $aa_1 + ba_2 + ca_3$, then only the ordering of the three real numbers a, b, and c matters. Indeed, in hindsight this choice was completely analogous to the choice we made (implicitly) in the case of $\mathfrak{sl}_2\mathbb{C}$ in choosing one of the two directions along the real line.

We said at the outset of this lecture that our goal was to arrive at a description of representations of $\mathfrak{sl}_3\mathbb{C}$ as complete as that for $\mathfrak{sl}_2\mathbb{C}$. We have now, certainly, as complete a description of the possible configurations of eigenvalues; but clearly much more is needed. Specifically, we should have

an existence and uniqueness theorem;

an explicit construction of each representations, analogous to the statement that every representation of $\mathfrak{sl}_2\mathbb{C}$ is a symmetric power of the standard; and

for the purpose of analyzing tensor products of representations of $\mathfrak{sl}_3\mathbb{C}$, we need a description not just of the set of eigenvalues, but of the multiplicities with which they occur.

(Note that the last question is one that has no analogue in the case of $\mathfrak{sl}_2\mathbb{C}$: in both cases, any irreducible representation is generated by taking a single eigenvector $v \in V_\alpha$ and pushing it around by elements of $\mathfrak{g}_\alpha$; but whereas in the previous case there was only one way to get from V_α to V_β—that is, by applying Y over and over again—in the present circumstance there will be more than one way of getting, for example, from V_α to $V_{\alpha+L_3-L_1}$; and these may yield independent eigenvectors.) This has been, however, already too long a lecture, and so we will defer these questions, along with all examples, to the next.

LECTURE 13

Representations of $\mathfrak{sl}_3\mathbb{C}$, Part II: Mainly Lots of Examples

In this lecture we complete the analysis of the irreducible representations of $\mathfrak{sl}_3\mathbb{C}$, culminating in §13.2 with the answers to all three of the questions raised at the end of the last lecture: we explicitly construct the unique irreducible representation with given highest weight, and in particular determine its multiplicities. The latter two sections correspond to §11.2 and §11.3 in the lecture on $\mathfrak{sl}_2\mathbb{C}$. In particular, §13.4, like §11.3, involves some projective algebraic geometry and may be skipped by those to whom this is unfamiliar.

§13.1. Examples

This lecture will be largely concerned with studying examples, giving constructions and analyzing tensor products of representations of $\mathfrak{sl}_3\mathbb{C}$. We start, however, by at least stating the basic existence and uniqueness theorem that provides the context for this analysis.

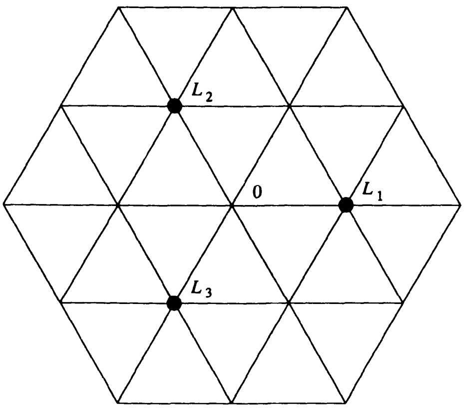
To state this, recall from the previous lecture than any irreducible, finite-dimensional representation of $\mathfrak{sl}_3\mathbb{C}$ has a vector, unique up to scalars, that is simultaneously an eigenvector for the subalgebra $\mathfrak{h}$ and killed by the three subspaces $\mathfrak{g}_{L_1-L_2}$, $\mathfrak{g}_{L_1-L_3}$, and $\mathfrak{g}_{L_2-L_3}$. We called such a vector a *highest weight vector* of the representation V; its associated eigenvalue will, of course, be called the *highest weight* of V. More generally, in any finite-dimensional representation W of $\mathfrak{sl}_3\mathbb{C}$, any vector $v \in W$ with these properties will be called a highest weight vector; we saw that it will generate an irreducible subrepresentation V of W. Finally, from the description given in the last lecture of the possible configurations of eigenvalues for a representation of $\mathfrak{sl}_3\mathbb{C}$, we see that any highest weight vector must lie in the $(\frac{1}{6})$-plane described by the inequalities $\langle H_{1,2}, L\rangle \geq 0$ and $\langle H_{2,3}, L\rangle \geq 0$, i.e., it must be of the form $(a+b)L_1 + bL_2 = aL_1 - bL_3$ for some pair of non-negative integers a and b. We can now state

Theorem 13.1. *For any pair of natural numbers a, b there exists a unique irreducible, finite-dimensional representation $\Gamma_{a,b}$ of $\mathfrak{sl}_3\mathbb{C}$ with highest weight $aL_1 - bL_3$.*

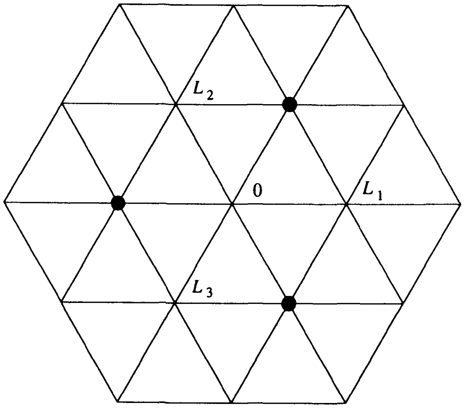
We will defer the proof of this theorem until the second section of this lecture, not so much because it is in any way difficult but simply because it is time to get to some examples. We will remark, however, that whereas in the case of $\mathfrak{sl}_2\mathbb{C}$ the analysis that led to the concept of highest weight vector immediately gave the uniqueness part of the analogous theorem, here to establish uniqueness we will be forced to resort to a more indirect trick. The proof of existence, by contrast, will be very much like that of the corresponding statement for $\mathfrak{sl}_2\mathbb{C}$: we will construct the representations $\Gamma_{a,b}$ out of the standard representation by multilinear algebra.

For the time being, though, we would like to apply the analysis of the previous lecture to some of the obvious representations of $\mathfrak{sl}_3\mathbb{C}$, partly to gain some familiarity with what goes on and partly in the hopes of seeing a general multilinear-algebraic construction.

We begin with the standard representation of $\mathfrak{sl}_3\mathbb{C}$ on $V \cong \mathbb{C}^3$. Of course, the eigenvectors for the action of $\mathfrak{h}$ are just the standard basis vectors e_1, e_2, and e_3; they have eigenvalues L_1, L_2, and L_3, respectively. The weight diagram for V is thus



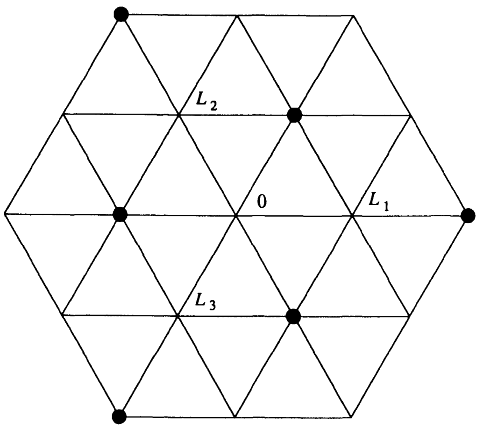
Next, consider the dual representation V^*. The eigenvalues of the dual of a representation of a Lie algebra are just the negatives of the eigenvalues of the original, so the diagram of V^* is



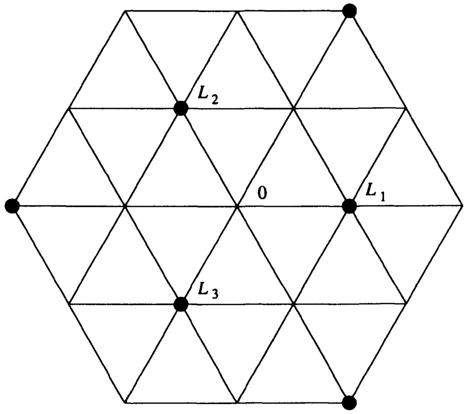
Alternatively, of course, we can just observe that the dual basis vectors e_i^* are eigenvectors with eigenvalues $-L_i$.

Note that while in the case of $\mathfrak{sl}_2\mathbb{C}$ the weights of any representation were symmetric about the origin, and correspondingly each representation was isomorphic to its dual, the same is not true here (that the diagrams for V and V^* look the same is a reflection of the fact that the two representations are carried into one another by an automorphism of $\mathfrak{sl}_3\mathbb{C}$, namely, the automorphism $X \mapsto -{}^tX$). Observe also that V^* is also isomorphic to the representation $\wedge^2 V$, whose weights are the pairwise sums of the distinct weights of V; and that likewise V is isomorphic as representation to $\wedge^2 V^*$.

Next, consider the degree 2 tensor products of V and V^*. Since the weights of the symmetric square of a representation are the pairwise sums of the weights of the original, the weight diagram of $\mathrm{Sym}^2 V$ will look like

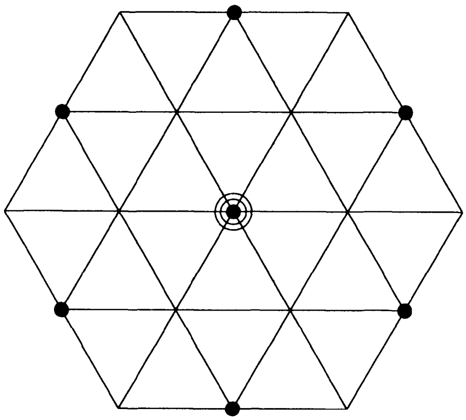


and likewise the symmetric square $\mathrm{Sym}^2 V^*$ has weights $\{-2L_i, -L_i - L_j\} = \{-2L_i - 2L_j, L_k\}$:



We see immediately from these diagrams that $\mathrm{Sym}^2 V$ and $\mathrm{Sym}^2 V^*$ are irreducible, since neither collection of weights is the union of two collections arising from representations of $\mathfrak{sl}_3\mathbb{C}$.

As for the tensor product $V \otimes V^*$, its weights are just the sums of the weights $\{L_i\}$ of V with those $\{-L_i\}$ of V^*, that is, the linear functionals $L_i - L_j$ (each occurring once, with weight vector $e_i \otimes e_j^*$) and 0 (occurring with multiplicity three, with weight vectors $e_i \otimes e_i^*$). We can represent these weights by the diagram



where the triple circle is intended to convey the fact that the weight space V_0 is three dimensional. By contrast with the last two examples, this representation is not irreducible: there is a linear map

$$V \otimes V^* \to \mathbb{C}$$

given simply by the contraction

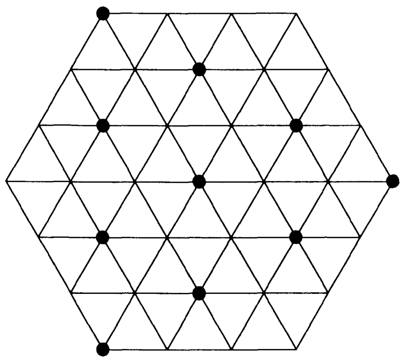
$$v \otimes u^* \mapsto \langle v, u^* \rangle = u^*(v)$$

(or, in terms of the identification $V \otimes V^* \cong \mathrm{Hom}(V, V)$, by the *trace*) that is a map of $\mathfrak{sl}_3\mathbb{C}$-modules (with $\mathbb{C}$ the trivial representation, of course). The kernel of this map is then the subspace of $V \otimes V^*$ of traceless matrices, which is just the adjoint representation of the Lie algebra $\mathfrak{sl}_3\mathbb{C}$ and is irreducible (we can see this either from our explicit description of the adjoint representation—for example, $E_{1,3}$ is the unique weight vector for $\mathfrak{h}$ killed by $\mathfrak{g}_{L_1-L_2}$, $\mathfrak{g}_{L_1-L_3}$, and $\mathfrak{g}_{L_2-L_3}$—or, if we take as known the fact that $\mathrm{SL}_3\mathbb{C}$ is simple, from the fact that a subrepresentation of the adjoint representation is an *ideal* in a Lie algebra, and exponentiates to a normal subgroup, cf. Exercise 8.43.)

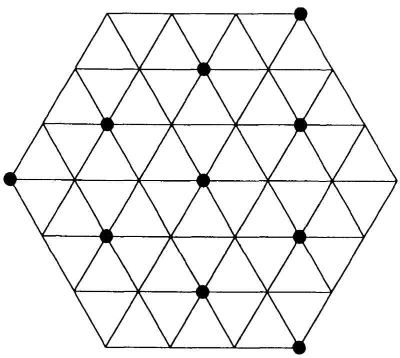
(Physicists call this adjoint representation of $\mathfrak{sl}_3\mathbb{C}$ (or SU(3)) the "eightfold way," and relate its decomposition to mesons and baryons. The standard representation V is related to "quarks" and V^* to "antiquarks." See [S-W], [Mack].)

(We note that, in general, if V is any faithful representation of a Lie algebra, the adjoint representation will appear as a subrepresentation of the tensor $V \otimes V^*$.)

Let us continue now with some of the triple tensor products of V and V^*, which will be the last specific cases we look at. To begin with, we have the symmetric cubes $\mathrm{Sym}^3 V$ and $\mathrm{Sym}^3 V^*$, with weight diagrams



and



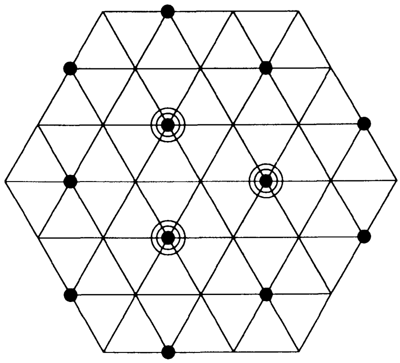
respectively. In general it is clear that, in terms of the description given in the preceding lecture of the possible weight diagrams of irreducible representations of $\mathfrak{sl}_3\mathbb{C}$, the symmetric powers of V and V^* will be exactly the representations with triangular, as opposed to hexagonal, diagrams.

It also follows from the above description and the fact that the weights of the symmetric powers $\mathrm{Sym}^n V$ occur with multiplicity 1 that $\mathrm{Sym}^n V$ and $\mathrm{Sym}^n V^*$ are all irreducible, i.e., we have, in the notation of Theorem 13.1,

$$\mathrm{Sym}^n V = \Gamma_{n,0} \quad \text{and} \quad \mathrm{Sym}^n V^* = \Gamma_{0,n}.$$

By way of notation, we will often write $\mathrm{Sym}^n V$ in place of $\Gamma_{n,0}$.

Consider now the mixed tensor $\mathrm{Sym}^2 V \otimes V^*$. Its weights are the sums of the weights of $\mathrm{Sym}^2 V$—that is, the pairwise sums of the L_i—with the weights of V^*; explicitly, these are $L_i + L_j - L_k$ and $2L_i - L_j$ (each occurring once) and the L_i themselves (each occurring three times, as $L_i + L_j - L_j$). Diagrammatically, the representation looks like



Now, we know right off the bat that this is not irreducible: we have a natural map

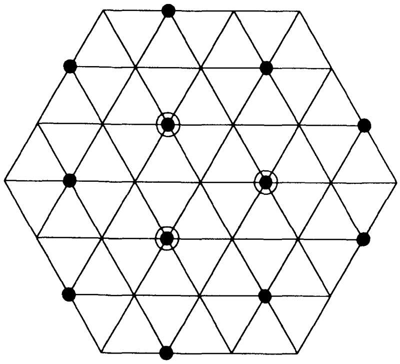
$$\imath\colon \mathrm{Sym}^2 V \otimes V^* \to V$$

given again by contraction, that is, by the map

$$vw \otimes u^* \mapsto \langle v, u^* \rangle \cdot w + \langle w, u^* \rangle \cdot v,$$

which is a map of $\mathfrak{sl}_3\mathbb{C}$-modules.[1] What does the kernel of this map look like? Of course, its weight diagram is

[1] Another way to see that $\mathrm{Sym}^2 V \otimes V^*$ is not irreducible is to observe that if a representation W is generated by a highest weight vector v of weight $2L_1 - L_3$, as $\mathrm{Sym}^2 V \otimes V^*$ must be if it is irreducible, the eigenvalue L_1 can be taken with multiplicity at most 2, the corresponding eigenspace being generated by $E_{2,1} \circ E_{3,2}(v)$ and $E_{3,2} \circ E_{2,1}(v)$.



and we know one other thing: certainly any vector in the weight space of $2L_1 - L_3$—that is to say, of course, any multiple of the vector $e_1^2 \otimes e_3^*$—is killed by $\mathfrak{g}_{L_1-L_2}$, $\mathfrak{g}_{L_1-L_3}$, and $\mathfrak{g}_{L_2-L_3}$, so that the kernel of ι will contain an irreducible representation $\Gamma = \Gamma_{2,1}$ with $2L_1 - L_3$ as its highest weight. Since Γ must then assume every weight of $\mathrm{Ker}(\iota)$, there are exactly two possibilities: either $\mathrm{Ker}(\iota) = \Gamma$, which assumes the weights L_i with multiplicity 2; or all the weights of Γ occur with multiplicity one and $\mathrm{Ker}(\iota) \cong \Gamma \oplus V$.

How do we settle this issue? There are at least three ways. To begin with, we can try to analyze directly the structure of the kernel of ι. An alternative approach would be to determine a priori with what multiplicities the weights of $\Gamma_{a,b}$ are taken. Certainly it is clear that a formula giving us the latter information will be tremendously valuable—it would for one thing clear up the present confusion instantly—and indeed there exist several such, one of which, the *Weyl character formula*, we will prove later in the book. (We will also prove the *Kostant multiplicity formula*, which can be applied to deduce directly the independence statement we arrive at below.) As a third possibility, we can identify the representations $\Gamma_{a,b}$ as Weyl modules and appeal to Lecture 6. Rather than invoke such general formulas at present, however, we will take the first approach here. This is straightforward: in terms of the notation we have been using, the highest weight vector for the representation $\Gamma \subset \mathrm{Sym}^2 V \otimes V^*$ is the vector $e_1^2 \otimes e_3^*$, and so the eigenspace $\Gamma_{L_1} \subset \Gamma$ with eigenvalue L_1 will be spanned by the images of this vector under the two compositions $E_{2,1} \circ E_{3,2}$ and $E_{3,2} \circ E_{2,1}$. These are, respectively,

$$\begin{aligned} E_{2,1} \circ E_{3,2}(e_1^2 \otimes e_3^*) &= E_{2,1}(E_{3,2}(e_1^2) \otimes e_3^* + e_1^2 \otimes E_{3,2}(e_3^*)) \\ &= E_{2,1}(-e_1^2 \otimes e_2^*) \\ &= -2(e_1 \cdot e_2) \otimes e_2^* + e_1^2 \otimes e_1^* \end{aligned}$$

and

$$\begin{aligned} E_{3,2} \circ E_{2,1}(e_1^2 \otimes e_3^*) &= E_{3,2}(E_{2,1}(e_1^2) \otimes e_3^* + e_1^2 \otimes E_{2,1}(e_3^*)) \\ &= E_{3,2}((2e_1 \cdot e_2) \otimes e_3^*) \\ &= 2(e_1 \cdot e_3) \otimes e_3^* - 2(e_1 \cdot e_2) \otimes e_2^*. \end{aligned}$$

Since these are independent, we conclude that the weight L_1 does occur in Γ with multiplicity 2, and hence that the kernel of ι is irreducible, i.e.,

$$\mathrm{Sym}^2 V \otimes V^* \cong \Gamma_{2,1} \oplus V.$$

§13.2. Description of the Irreducible Representations

At this point, rather than go on with more examples we should state some of the general principles that have emerged so far. The first and most important (though pretty obvious) is the basic

Observation 13.2. *If the representations V and W have highest weight vectors v and w with weights α and β, respectively, then the vector $v \otimes w \in V \otimes W$ is a highest weight vector of weight $\alpha + \beta$.*

Of course, there are numerous generalizations of this: the vector $v^n \in \mathrm{Sym}^n V$ is a highest weight vector of weight $n\alpha$, etc.[2] Just the basic statement above, however, enables us to give the

Proof of Theorem 13.1. First, the existence statement follows immediately from the observation: the representation $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ will contain an irreducible subrepresentation $\Gamma_{a,b}$ with highest weight $aL_1 - bL_3$.

The uniqueness part is only slightly harder (if less explicit): Given irreducible representations V and W with highest weight α, let $v \in V$ and $w \in W$ be highest weight vectors with weight α. Then (v, w) is again a highest weight vector in the representation $V \oplus W$ with highest weight α; let $U \subset V \oplus W$ be the irreducible subrepresentation generated by (v, w). The projection maps $\pi_1: U \to V$ and $\pi_2: U \to W$, being nonzero maps between irreducible representations of $\mathfrak{sl}_3\mathbb{C}$, must be isomorphisms, and we deduce that $V \cong W$. □

Exercise 13.3*. Let $\mathbb{S}_\lambda$ be the Schur functor introduced in Lecture 6. What can you say about the highest weight vectors in the representation $\mathbb{S}_\lambda(V)$ obtained by applying it to a given representation V?

To continue our discussion of tensor products like $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ in general, as we indicated we would like to make more explicit the construction of the representation $\Gamma_{a,b}$, which we know to be lying in $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$. To begin with, we have in general a contraction map

$$\iota_{a,b}: \mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^* \to \mathrm{Sym}^{a-1} V \otimes \mathrm{Sym}^{b-1} V^*$$

analogous to the map ι introduced above; we can describe this map either (in fancy language) as the dual of the map from $\mathrm{Sym}^{a-1} V \otimes \mathrm{Sym}^{b-1} V^*$ to $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ given by multiplication by the identity element in

[2] One slightly less obvious statement is this: if the weights of V are $\alpha_1, \alpha_2, \alpha_3 \ldots$ with $l(\alpha_1) > l(\alpha_2) > \ldots$, then $\bigwedge^n V$ possesses a highest weight vector weight $\alpha_1 + \cdots + \alpha_n$. Note that since the ordering of the α_i may in fact depend on the choice of l (even with the restriction $a > b > c$ on the coefficients of l as above), this may in some cases imply the existence of several subrepresentations of $\bigwedge^n V$.

$V \otimes V^* = \mathrm{Hom}(V, V)$; or, concretely, by sending

$$(v_1 \cdot \ldots \cdot v_a) \otimes (v_1^* \cdot \ldots \cdot v_b^*)$$
$$\mapsto \sum \langle v_i, v_j^* \rangle (v_1 \cdot \ldots \cdot \hat{v}_i \cdot \ldots \cdot v_a) \otimes (v_1^* \cdot \ldots \cdot \hat{v}_j^* \cdot \ldots \cdot v_b^*).$$

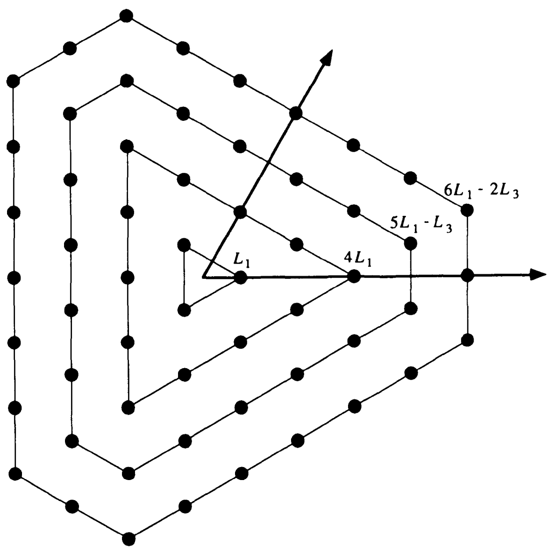
Clearly this map is surjective, and, since the target does not have eigenvalue $aL_1 - bL_3$, the subrepresentation $\Gamma_{a,b} \subset \mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ must lie in the kernel. In fact, we have, just as in the case of $\mathrm{Sym}^2 V \otimes V^*$ above,

Claim 13.4. *The kernel of the map $\iota_{a,b}$ is the irreducible representation $\Gamma_{a,b}$.*

We will defer the proof of this for a moment and consider some of its consequences. To begin with, we can deduce from this assertion the complete decomposition of $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$: we must have (if, say, $b \le a$)

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^* = \bigoplus_{i=0}^{b} \Gamma_{a-i,b-i}. \tag{13.5}$$

Since we know, a priori, all the multiplicities of the eigenvalues of the tensor product $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$, this will, in turn, determine (inductively at least) all the multiplicities of the representations $\Gamma_{a,b}$. In fact, the answer turns out to be very nice. To express it, observe first that if $a \ge b$, the weight diagram of either $\Gamma_{a,b}$ or $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ looks like a sequence of b shrinking concentric (not in general regular) hexagons H_i with vertices at the points $(a-i)L_1 - (b-i)L_3$ for $i = 0, 1, \ldots, b-1$, followed (after the shorter three sides of the hexagon have shrunk to points) by a sequence of $[(a-b)/3] + 1$ triangles T_j with vertices at the points $(a-b-3j)L_1$ for $j = 0, 1, \ldots, [(a-b)/3]$ (it will be convenient notationally to refer to T_0 as H_b occasionally). Diagram (13.6) shows the picture of the weights of $\mathrm{Sym}^6 V \otimes \mathrm{Sym}^2 V^*$:



(13.6)

(Note that by the decomposition (13.5), the weights of the highest weight vectors in $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ will be $aL_1 - bL_3, (a-1)L_1 - (b-1)L_3, \ldots, (a-b)L_1$, as shown in the diagram.)

An examination of the representation $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ shows that it has multiplicity $(i+1)(i+2)/2$ on the hexagon H_i, and then a constant multiplicity $(b+1)(b+2)/2$ on all the triangles T_j; and it follows from the decomposition (13.5), in general, that the representation $\Gamma_{a,b}$ has multiplicity $(i+1)$ on H_i and $b+1$ on T_j. In English, *the multiplicities of* $\Gamma_{a,b}$ *increase by one on each of the concentric hexagons of the eigenvalue diagram and are constant on the triangles.* Note in particular that the description of $\Gamma_{2,1}$ in the preceding section is a special case of this.

PROOF OF CLAIM 13.4. We remark first that the claim will be implied by the Weyl character formula or by the description via Weyl's construction in Lecture 15; so the reader who wishes to can skip the following without dire consequences to the logical structure of the book. Otherwise, observe first that the claim is equivalent to asserting the decomposition (13.5); this, in turn, is equivalent to the statement that the representation $W = \mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ has exactly $b+1$ irreducible components (still assuming $a \geq b$). The irreducible factors in a representation correspond to the highest weight vectors in the representation up to scalars; so in sum the claim is equivalent to the assertion that *the eigenspace* W_α *of* $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ *contains a unique highest weight vector (up to scalars) if* α *is of the form* $(a-i)L_1 - (b-i)L_3$ *for* $i \leq b$, *and none otherwise*; this is what we shall prove.

To begin with, the "none otherwise" part of the statement follows (given the other) just from looking at the diagram: if, for example, any of the eigenspaces W_α corresponding to a point α on a hexagon H_i (other than the vertex $(a-i)L_1 - (b-i)L_3$ of H_i) possessed a highest weight vector, the multiplicity of α in W would be strictly greater than of $(a-i)L_1 - (b-i)L_3$, which we know is not the case; similarly, the fact that the multiplicities of W in the triangular part of the eigenvalue diagram are constant implies that there can be no highest weight vectors with eigenvalue on a T_j for $j \geq 1$. Thus, we just have to check that the weight spaces W_α for $\alpha = (a-i)L_1 - (b-i)L_3$ contain only the one highest weight vector we know is there; and we do this by explicit calculation.

To start, for any monomial index $I = (i_1, i_2, i_3)$ of degree $\sum i_\gamma = i$, we denote by $e^I \in \mathrm{Sym}^i V$ the corresponding monomial $\prod (e_\gamma^{i_\gamma})$ and define $(e^*)^I \in \mathrm{Sym}^i V^*$ similarly. We can then write any element of the weight space $W_{(a-i)L_1-(b-i)L_3}$ of $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^*$ as

$$v = \sum c_I \cdot (e_1^{a-i} \cdot e^I) \otimes ((e_3^*)^{b-i} \cdot (e^*)^I).$$

In these terms, it is easy to write down the action of the two operators $E_{1,2}$

and $E_{2,3}$. First, $E_{1,2}$ kills both $e_1 \in V$ and $e_3^* \in V^*$, so that we have

$$\begin{aligned} E_{1,2}((e_1^{a-i} \cdot e^I) &\otimes ((e_3^*)^{b-i} \cdot (e^*)^I)) \\ &= i_2(e_1^{a-i} \cdot e^{I'}) \otimes ((e_3^*)^{b-i} \cdot (e^*)^I) \\ &\quad - i_1(e_1^{a-i} \cdot e^I) \otimes ((e_3^*)^{b-i} \cdot (e^*)^{I''}), \end{aligned}$$

where $I' = (i_1 + 1, i_2 - 1, i_3)$ and $I'' = (i_1 - 1, i_2 + 1, i_3)$ (and we adopt the convention that $e^I = 0$ if $i_\gamma < 0$ for any γ). It follows that *the vector v above is in the kernel of $E_{1,2}$ if and only if the coefficients c_I satisfy $i_2 c_I = (i_1 + 1)c_{I'}$*; and by the analogous calculation that v is in the kernel of $E_{2,3}$ if and only if $i_3 c_I = (i_2 + 1)c_J$ whenever the indices I and J are related by $j_1 = i_1$, $j_2 = i_2 + 1$, and $j_3 = i_3 - 1$. These conditions are equivalent to saying that the numbers $i_1!i_2!i_3!c_I$ are independent of I. We see, in other words, that v is a highest weight vector if and only if all the coefficients c_I are equal to $c/i_1!i_2!i_3!$ for some constant c. □

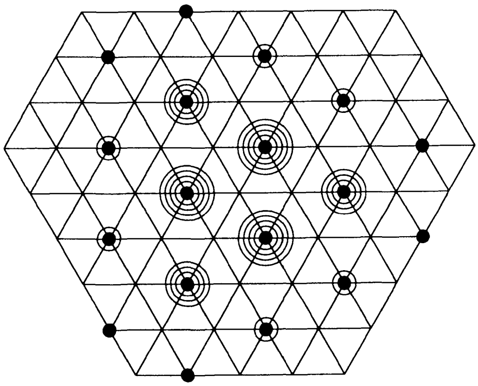
§13.3. A Little More Plethysm

We would like to consider here, as we did in the case of $\mathfrak{sl}_2\mathbb{C}$ in Lecture 11, how the tensor products and powers of the representations we have described decompose. We start with one general remark: given our knowledge of the eigenvalue diagrams of the irreducible representations of $\mathfrak{sl}_3\mathbb{C}$ (*with* multiplicities), there can be no possible ambiguity about the decomposition of any representation U given as the tensor product of representations whose eigenvalue diagrams are known. Indeed, we have an algorithm for determining the components of that decomposition, as follows:

1. Write down the eigenvalue decomposition of U.
2. Find the eigenvalue $\alpha = aL_1 - bL_3$ appearing in this diagram for which the value of $l(\alpha)$ is maximal.
3. We now know that U will contain a copy of the irreducible representation $\Gamma_\alpha = \Gamma_{a,b}$, i.e., $U \cong \Gamma_\alpha \oplus U'$ for some U'. Since we also know the eigenvalue diagram of Γ_α, we can thus write down the eigenvalue diagram of U' as well.
4. Repeat this process for U'.

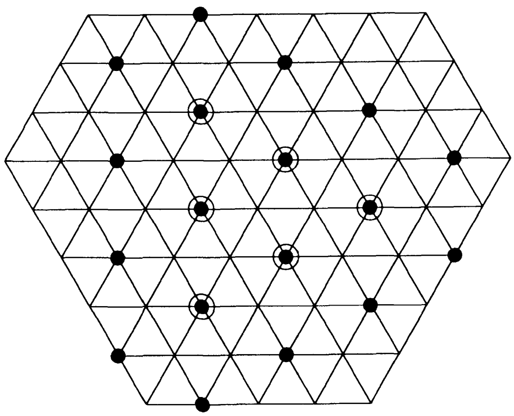
To see how this goes in practice, consider some examples of tensor products of the basic irreducible representations described so far. We have already seen how the tensor products of the symmetric powers of the standard representation V of $\mathfrak{sl}_3\mathbb{C}$ and symmetric powers of its dual decompose; let us look now at an example of a more general tensor product of irreducible representations: say V itself and the representation $\Gamma_{2,1}$. We start by writing down the weights of the tensor product: since $\Gamma_{2,1}$ has weights $2L_i - L_j$, $L_i + L_j - L_k$, and L_i

(taken twice) and V has weights L_i, the tensor product will have weights $3L_i - L_j$, $2L_i + L_j - L_k$ (taken twice), $2L_i$ (taken four times), and $L_i + L_j$ (taken five times). The diagram is thus

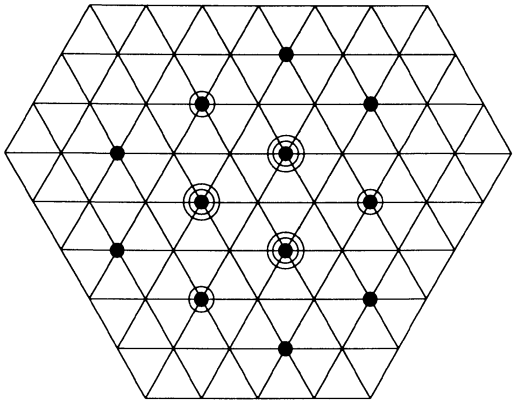


(One thing we may deduce from this diagram is that we are soon going to need a better system for presenting the data of the weights of a representation. In the future, we may simply draw one sector of the plane, and label weights with numbers to indicate multiplicities.)

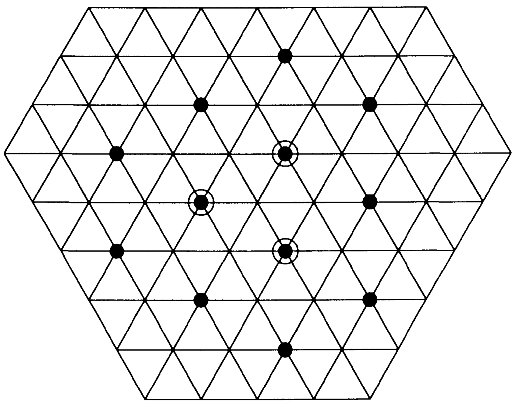
We know right off the bat that the tensor product $V \otimes \Gamma_{2,1}$ contains a copy of the irreducible representation $\Gamma_{3,1}$ with highest weight $3L_1 - L_3$. By what we have said, the weight diagram of $\Gamma_{3,1}$ is



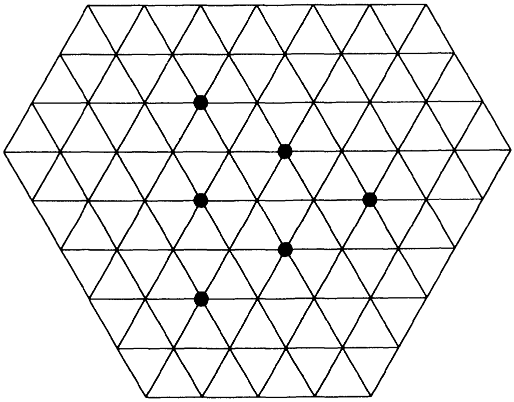
so the complement of $\Gamma_{3,1}$ in the tensor product $V \otimes \Gamma_{2,1}$ will look like



One obvious highest weight in this representation is $2L_1 + L_2 - L_3 = L_1 - 2L_3$, so that the tensor product will contain a copy of the irreducible representation $\Gamma_{1,2}$ as well; since this has weight diagram



the remaining part of the tensor product will have weight diagram

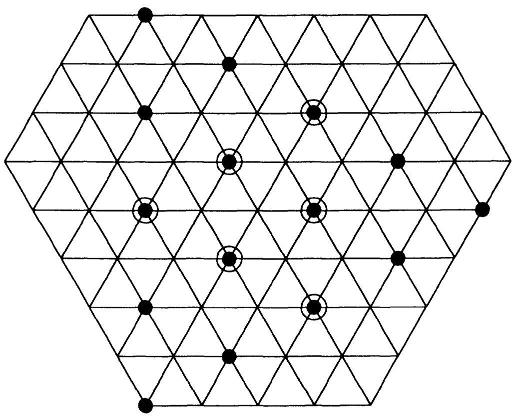


which we recognize as the weight diagram of the symmetric square $\mathrm{Sym}^2 V = \Gamma_{2,0}$ of the standard representation. We have, thus,

$$V \otimes \Gamma_{2,1} = \Gamma_{3,1} \oplus \Gamma_{1,2} \oplus \Gamma_{2,0}. \tag{13.7}$$

Exercise 13.8*. Find the decomposition into irreducible representations of the tensor products $V \otimes \Gamma_{1,1}$, $V \otimes \Gamma_{1,2}$ and $V \otimes \Gamma_{3,1}$. Can you find a general pattern to the outcomes?

As in the case of $\mathfrak{sl}_2\mathbb{C}$, the next thing to look at are the tensor powers—symmetric and exterior—of representations other than the standard; we look first at tensors of the symmetric square $W = \mathrm{Sym}^2 V$. First, consider the symmetric square $\mathrm{Sym}^2 W = \mathrm{Sym}^2(\mathrm{Sym}^2 V))$. We know the diagram for $\mathrm{Sym}^2 W$; it is



Now, there is only one possible decomposition of a representation whose eigenvalue diagram looks like this: we must have

$$\mathrm{Sym}^2(\mathrm{Sym}^2 V)) \cong \mathrm{Sym}^4 V \oplus \mathrm{Sym}^2 V^*.$$

Indeed, the presence of the $\mathrm{Sym}^4 V$ factor is clear: there is an obvious map

$$\varphi\colon \mathrm{Sym}^2(\mathrm{Sym}^2 V)) \to \mathrm{Sym}^4 V$$

obtained simply by multiplying out. The identification of the kernel of this map with the representation $\mathrm{Sym}^2 V^*$ is certainly less obvious, but can still be made explicit. We can identify V^* with $\wedge^2 V$ as we saw, and then define a map

$$\tau\colon \mathrm{Sym}^2(\wedge^2 V) \to \mathrm{Sym}^2(\mathrm{Sym}^2 V))$$

by sending the generator $(u \wedge v)\cdot(w \wedge z) \in \mathrm{Sym}^2(\wedge^2 V)$ to the element $(u \cdot w)\cdot(v \cdot z) - (u \cdot z)\cdot(v \cdot w) \in \mathrm{Sym}^2(\mathrm{Sym}^2 V)$, which is clearly in the kernel of φ.

Exercise 13.9. Verify that this map is well defined and that it extends linearly to an isomorphism of $\mathrm{Sym}^2(\wedge^2 V)$ with $\mathrm{Ker}(\varphi)$.

Exercise 13.10. Apply the techniques above to show that the representation $\wedge^2(\mathrm{Sym}^2 V)$ is isomorphic to $\Gamma_{2,1}$.

Exercise 13.11. Apply the same techniques to determine the irreducible factors of the representation $\wedge^3(\mathrm{Sym}^2 V)$. Note: we will return to this example in Exercise 13.22.

Exercise 13.12. Find the decomposition into irreducibles of the representations $\mathrm{Sym}^2(\mathrm{Sym}^3 V)$ and $\mathrm{Sym}^3(\mathrm{Sym}^2 V)$ (observe in particular that Hermite reciprocity has bitten the dust). Describe the projection maps to the various factors. Note: we will describe these examples further in the following section.

§13.4. A Little More Geometric Plethysm

Just as in the case of $\mathfrak{sl}_2\mathbb{C}$, some of these identifications can also be seen in geometric terms. To do this, recall from §11.3 the definition of the *Veronese embedding*: if $\mathbb{P}^2 = \mathbb{P}V^*$ is the projective space of one-dimensional subspaces of V^*, there is then a natural embedding of $\mathbb{P}^2$ in the projective space $\mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}^2 V^*)$, obtained simply by sending the point $[v^*] \in \mathbb{P}^2$ corresponding to the vector $v^* \in V^*$ to the point $[v^{*2}] \in \mathbb{P}(\mathrm{Sym}^2 V^*)$ associated to the vector $v^{*2} = v^* \cdot v^* \in \mathrm{Sym}^2 V^*$. The image $S \subset \mathbb{P}^5$ is called the *Veronese surface.* As in the case of the rational normal curves discussed in Lecture 11, it is not hard to see that the group of automorphisms of $\mathbb{P}^5$ carrying S into itself is exactly the group $\mathrm{PGL}_3\mathbb{C}$ of automorphisms of $S = \mathbb{P}^2$.

Now, a quadratic polynomial in the homogeneous coordinates of the space $\mathbb{P}(\mathrm{Sym}^2 V^*) \cong \mathbb{P}^5$ will restrict to a quartic polynomial on the Veronese surface $S = \mathbb{P}V^*$, which corresponds to the natural evaluation map φ of the preceding section; the kernel of this map is thus the vector space of quadratic polynomials in $\mathbb{P}^5$ vanishing on the Veronese surface S, on which the group of automorphisms of $\mathbb{P}^5$ carrying S to itself obviously acts. Now, for any pair of points $P = [u^*]$, $Q = [v^*] \in S$, it is not hard to see that the cone over the Veronese surface with vertex the line $\overline{PQ} \subset \mathbb{P}^5$ (that is, the union of the 2-planes $\overline{PQR}$ as R varies over the surface S) will be a quadric hypersurface in $\mathbb{P}^5$ containing the Veronese surface; sending the generator $u^* \cdot v^* \in \mathrm{Sym}^2 V^*$ to this quadric hypersurface will then define an isomorphism of the space of such quadrics with the projective space associated to $\mathrm{Sym}^2 V^*$.

Exercise 13.13. Verify the statements made in the last paragraph: that the union of the $\overline{PQR}$ is a quadric hypersurface and that this extends to a linear isomorphism $\mathbb{P}(\mathrm{Sym}^2 V^*) \cong \mathbb{P}(\mathrm{Ker}(\varphi))$. Verify also that this isomorphism coincides with the one given in Exercise 13.9.

There is another way of representing the Veronese surface that will shed some light on this kernel. If, in terms of some coordinates e_i on V^*, we think of $\mathrm{Sym}^2 V^*$ as the vector space of symmetric 3×3 matrices, then the Veronese surface is just the locus, in the associated projective space, of rank 1 matrices up to scalars, i.e., in terms of homogeneous coordinates $Z_{i,j} = e_i \cdot e_j$ on $\mathbb{P}^5$,

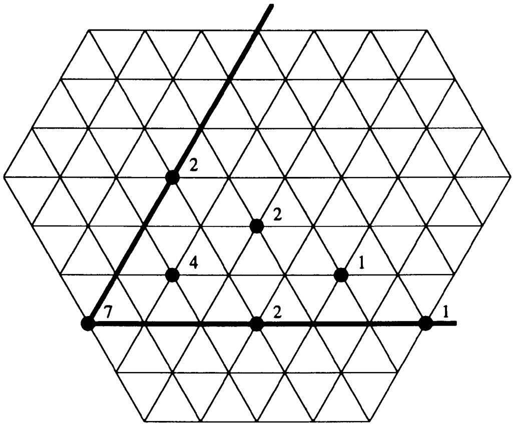
$$S = \left\{ [Z]\colon \mathrm{rank}\begin{pmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} \\ Z_{1,2} & Z_{2,2} & Z_{2,3} \\ Z_{1,3} & Z_{2,3} & Z_{3,3} \end{pmatrix} = 1 \right\}.$$

The vector space of quadratic polynomials vanishing on S is then generated by the 2×2 minors of the matrix $(Z_{i,j})$; in particular, for any pair of linear combinations of the rows and pair of linear combinations of the columns we get a 2×2 matrix whose determinant vanishes on S.

Exercise 13.14. Show that this is exactly the isomorphism $\mathrm{Sym}^2(\wedge^2 V) \cong \mathrm{Ker}(\varphi)$ described above.

We note in passing that if indeed the space of quadrics containing the Veronese surface, with the action of the group $\mathrm{PGL}_3\mathbb{C}$ of motions of $\mathbb{P}^5$ preserving S, is the projectivization of the representation $\mathrm{Sym}^2 V^*$, then it must contain its own Veronese surface, i.e., there must be a surface $T = \mathbb{P}(V^*) \subset \mathbb{P}(\mathrm{Ker}(\varphi))$ invariant under this action. This turns out to be just the set of *quadrics of rank* 3 containing the Veronese, that is, the quadrics whose singular locus is a 2-plane. In fact, the 2-plane will be the tangent plane to S at a point, giving the identification $T = S$.

Let us consider one more example of this type, namely, the symmetric cube $\mathrm{Sym}^3(\mathrm{Sym}^2 V)$). (We promise we will stop after this one.) As before, it is easy to write down the eigenvalues of this representation; they are just the triple sums of the eigenvalues $\{2L_i, L_i + L_j\}$ of $\mathrm{Sym}^2 V$. The diagram (we will draw here only one-sixth of the plane and indicate multiplicities with numbers rather than rings) thus looks like



from which we see what the decomposition must be: as representations we have

$$\mathrm{Sym}^3(\mathrm{Sym}^2 V)) \cong \mathrm{Sym}^6 V \oplus \Gamma_{2,2} \oplus \mathbb{C}. \tag{13.15}$$

As before, the map to the first factor is just the obvious one; it is the identification of the kernel that is intriguing, and especially the identification of the last factor.

To see what is going on here, we should look again at the geometry of the Veronese surface $S \subset \mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}^2 V^*)$. The space $\mathrm{Sym}^3(\mathrm{Sym}^2 V))$ is just the space of homogeneous cubic polynomials on the ambient space $\mathbb{P}^5$, and as before the map to the first factor of the right-hand side of (13.15) is just the restriction, so that the last two factors of (13.15) represent the vector space $I(S)_3$ of cubic polynomials vanishing on S. Note that we could in fact prove (13.15) without recourse to eigenvalue diagrams from this: since the ideal of the Veronese surface is generated by the vector space $I(S)_2$ of quadratic polynomials vanishing on it, we have a surjective map

$$I(S)_2 \otimes W = \mathrm{Sym}^2 V^* \otimes \mathrm{Sym}^2 V \to I(S)_3.$$

But we already know how the left hand side decomposes: we have

$$\mathrm{Sym}^2 V^* \otimes \mathrm{Sym}^2 V = \Gamma_{2,2} \oplus \Gamma_{1,1} \oplus \mathbb{C}, \tag{13.16}$$

so that $I(S)_3$ must be a partial direct sum of these three irreducible representations; by dimension considerations it can only be $\Gamma_{2,2} \oplus \mathbb{C}$.

This, in turn, tells us how to make the isomorphism (13.15) explicit (assuming we want to): we can define a map

$$\mathrm{Sym}^2(\wedge^2 V) \otimes \mathrm{Sym}^2 V \to \mathrm{Sym}^3(\mathrm{Sym}^2 V)$$

by sending

$$(u \wedge v)\cdot(w \wedge z) \otimes (s \cdot t) \mapsto ((u \cdot w)\cdot(v \cdot z) - (u \cdot z)\cdot(v \cdot w))\cdot(s \cdot t)$$

and then just check that this gives an isomorphism of $\Gamma_{2,2} \oplus \mathbb{C} \subset \mathrm{Sym}^2 V^* \otimes \mathrm{Sym}^2 V$ with the kernel of projection on the first factor of the right-hand side of (13.15).

What is really most interesting in this whole situation, though, is the trivial summand in the expression (13.15). To say that there is such a summand is to say that *there exists a cubic hypersurface X in $\mathbb{P}^5$ preserved under all automorphisms of $\mathbb{P}^5$ carrying S to itself.* Of course, we have already run into this one: it is the determinant of the 3×3 matrix $(Z_{i,j})$ introduced above. To express this more intrinsically, if we think of the Veronese as the set of rank 1 tensors in $\mathrm{Sym}^2 V^*$, it is just the set of tensors of rank 2 or less. This, in turn, yields another description of X: since a rank 2 tensor is just one that can be expressed as a linear combination of two rank 1 tensors, we see that X is the famous *chordal variety* of the Veronese surface: it is the union of the chords to S, and at the same time the union of all the tangent planes to S.

Exercise 13.17. Show that the only symmetric powers of $\mathrm{Sym}^2 V$ that possess trivial summands are the powers $\mathrm{Sym}^{3k}(\mathrm{Sym}^2 V))$ divisible by 3, and that the unique trivial summand in this is just the kth power of the trivial summand of $\mathrm{Sym}^3(\mathrm{Sym}^2 V))$.

Exercise 13.18. Given the isomorphism of the projectivization of the vector space $I(S)_2$—that is, the projective space of quadric hypersurfaces containing the Veronese surface—with $\mathbb{P}(\mathrm{Sym}^2 V^*)$, find the unique cubic hypersurface in $I(S)_2$ invariant under the action of $\mathrm{PGL}_3\mathbb{C}$.

Exercise 13.19. Analyze the representation $\mathrm{Sym}^2(\mathrm{Sym}^3 V))$ of $\mathfrak{sl}_3\mathbb{C}$. Interpret the direct sum factors in terms of the geometry of the Veronese embedding of $\mathbb{P}V^* = \mathbb{P}^2$ in $\mathbb{P}(\mathrm{Sym}^3 V^*) = \mathbb{P}^9$.

Exercise 13.20*. Show that the representations $\mathrm{Sym}^4(\mathrm{Sym}^3 V))$ and $\mathrm{Sym}^6(\mathrm{Sym}^3 V))$ contain trivial summands, and that the representation $\mathrm{Sym}^{12}(\mathrm{Sym}^3 V))$ contains two. Interpret these.

Exercise 13.21. Apply the techniques above to show that the representation $\wedge^2(\mathrm{Sym}^2 V)$ is isomorphic to $\Gamma_{2,1}$.

Exercise 13.22*. Apply the techniques above to analyze the representation $\wedge^3(\mathrm{Sym}^2 V)$, and in particular to interpret its decomposition into irreducible representations.

Exercise 13.23. If $\mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}^2 V^*)$ is the ambient space of the Veronese surface, the Grassmannian $\mathbb{G}(2, 5)$ of 2-planes in $\mathbb{P}^5$ naturally embeds in the projective space $\mathbb{P}(\wedge^3(\mathrm{Sym}^2 V))$. Describe, in terms of the decomposition in the preceding exercise, the span of the locus of tangent 2-planes to the

Veronese, and the span of the locus of 2-planes in $\mathbb{P}^5$ spanned by the images in S of lines in $\mathbb{P}V^*$.

Exercise 13.24*. Show that the unique closed orbit of the action of $SL_3\mathbb{C}$ on the representation $\Gamma_{a,b}$ is either isomorphic to $\mathbb{P}^2$ (embedded as the Veronese surface) if either a or b is zero, or to the incidence correspondence

$$\Sigma = \{(p, l) : p \in l\} \subset \mathbb{P}^2 \times \mathbb{P}^{2*}$$

if neither a or b is zero.

PART III

THE CLASSICAL LIE ALGEBRAS AND THEIR REPRESENTATIONS

As we indicated at the outset, the analysis we have just carried out of the structure of $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$ and their representations carries over to other semisimple complex Lie algebras. In Lecture 14 we codify this structure, using the pattern of the examples we have worked out so far to give a model for the analysis of arbitrary semisimple Lie algebras and stating some of the most important facts that are true in general. As usual, we postpone proofs of many of these facts until Part IV and the Appendices, the main point here being to introduce a unifying approach and language. The facts themselves will all be seen explicitly on a case-by-case basis for the classical Lie algebras $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{sp}_{2n}\mathbb{C}$, and $\mathfrak{so}_n\mathbb{C}$, which are studied in some detail in Lectures 15–20.

Most of the development follows the outline we developed in Lectures 11–13, the main goal being to describe the irreducible representations as explicitly as we can, and to see the decomposition of naturally occurring representations, both algebraically and geometrically. While most of the representations are found inside tensor powers of the standard representations, for the orthogonal Lie algebras this only gives half of them, and one needs new methods to construct the other "spin" representations. This is carried out using Clifford algebras in Lecture 20.

We also make the tie with Weyl's construction of representations of $GL_n\mathbb{C}$ from Lecture 6, which arose from the representation theory of the symmetric groups. We show in Lecture 15 that these are the irreducible representations of $\mathfrak{sl}_n\mathbb{C}$; in Lecture 17 we show how to use them to construct the irreducible representations of the symplectic Lie algebras, and in Lecture 19 to give the nonspin representation of the orthogonal Lie algebras. These give useful descriptions of the irreducible representations, and powerful methods for decomposing other representations, but they are not necessary for the logical progression of the book, and many of these decompositions can also be deduced from the Weyl character formula which we will discuss in Part IV.

LECTURE 14

The General Setup: Analyzing the Structure and Representations of an Arbitrary Semisimple Lie Algebra

This is the last of the four central lectures; in the body of it, §14.1, we extract from the examples of §11–13 the basic algorithm for analyzing a general semisimple Lie algebra and its representations. It is this algorithm that we will spend the remainder of Part III carrying out for the classical algebras, and the reader who finds the general setup confusing may wish to read this lecture in parallel with, for example, Lectures 15 and 16. In particular, §14.2 is less clearly motivated by what we have worked out so far; the reader may wish to skim it for now and defer a more thorough reading until after going through some more of the examples of Lectures 15–20.

§14.1. Analyzing Simple Lie Algebras in General

We said at the outset of Lecture 12 that once the analysis of the representations of $\mathfrak{sl}_3\mathbb{C}$ was understood, the analysis of the representations of any semisimple Lie algebra would be clear, at least in broad outline. Here we would like to indicate how that analysis will go in general, by providing an essentially algorithmic procedure for describing the representations of an arbitrary complex semisimple Lie algebra $\mathfrak{g}$. The process we give here is directly analogous, step for step, to that carried out in Lecture 12 for $\mathfrak{sl}_3\mathbb{C}$; the only difference is one change in the order of steps: having seen in the case of $\mathfrak{sl}_3\mathbb{C}$ the importance of the "distinguished" subalgebras $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C} \subset \mathfrak{g}$ and the corresponding distinguished elements $H_\alpha \in \mathfrak{s}_\alpha \subset \mathfrak{h}$, we will introduce them earlier here.

Step 0. *Verify that your Lie algebra is semisimple*; if not, none of the following will work (but see Remark 14.3). If your Lie algebra is not semisimple, pass as indicated in Lecture 9 to its semisimple part; a knowledge of the representations of this quotient algebra may not tell you everything about the representations of the original, but it will at least tell you about the irreducible representations.

Step 1. *Find an abelian subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *acting diagonally.* This is of course the analogue of looking at the specific element H in $\mathfrak{sl}_2\mathbb{C}$ and the subalgebra $\mathfrak{h}$ of diagonal matrices in the case of $\mathfrak{sl}_3\mathbb{C}$; in general, to serve an analogous function it should be an abelian subalgebra that acts diagonally on one faithful (and hence, by Theorem 9.20, on any) representation of $\mathfrak{g}$. Moreover, in order that the restriction of a representation V of $\mathfrak{g}$ to $\mathfrak{h}$ carry the greatest possible information about V, $\mathfrak{h}$ should clearly be maximal among abelian, diagonalizable subalgebras; such a subalgebra is called a *Cartan subalgebra.*

It may seem that this step is somewhat less than algorithmic; in particular, while it is certainly possible to tell when a subalgebra of a given Lie algebra is abelian, and when it is diagonalizable, it is not clear how to tell whether it is maximal with respect to these properties. This defect will, however, be largely cleared up in the next step (see Remark 14.3).

Step 2. *Let* $\mathfrak{h}$ *act on* $\mathfrak{g}$ *by the adjoint representation, and decompose* $\mathfrak{g}$ *accordingly.* By the choice of $\mathfrak{h}$, its action on any representation of $\mathfrak{g}$ will be diagonalizable; applying this to the adjoint representation we arrive at a direct sum decomposition, called a *Cartan decomposition,*

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus \mathfrak{g}_\alpha\right), \tag{14.1}$$

where the action of $\mathfrak{h}$ preserves each $\mathfrak{g}_\alpha$ and acts on it by scalar multiplication by the linear functional $\alpha \in \mathfrak{h}^*$; that is, for any $H \in \mathfrak{h}$ and $X \in \mathfrak{g}_\alpha$ we will have

$$\mathrm{ad}(H)(X) = \alpha(H) \cdot X.$$

The second direct sum in the expression (14.1) is over a finite set of eigenvalues $\alpha \in \mathfrak{h}^*$; these eigenvalues—in the language of Lecture 12, the *weights of the adjoint representation*—are called the *roots* of the Lie algebra and the corresponding subspaces $\mathfrak{g}_\alpha$ are called the *root spaces.* Of course, $\mathfrak{h}$ itself is just the eigenspace for the action of $\mathfrak{h}$ corresponding to the eigenvalue 0 (see Remark 14.3 below); so that in some contexts—such as the following paragraph, for example—it will be convenient to adopt the convention that $\mathfrak{g}_0 = \mathfrak{h}$; but we do not usually count $0 \in \mathfrak{h}^*$ as a root. The set of all roots is usually denoted $R \subset \mathfrak{h}^*$.

As in the previous cases, we can picture the structure of the Lie algebra in terms of the diagram of its roots: by the fundamental calculation of §11.1 and Lecture 12 (which we will not reproduce here for the fourth time) we see that the adjoint action of $\mathfrak{g}_\alpha$ carries the eigenspace $\mathfrak{g}_\beta$ into another eigenspace $\mathfrak{g}_{\alpha+\beta}$.

There are a couple of things we can anticipate about how the configuration of roots (and the corresponding root spaces) will look. We will simply state them here as

Facts 14.2

(i) *each root space* $\mathfrak{g}_\alpha$ *will be one dimensional.*
(ii) R *will generate a lattice* $\Lambda_R \subset \mathfrak{h}^*$ *of rank equal to the dimension of* $\mathfrak{h}$.

(iii) *R is symmetric about the origin, i.e., if $\alpha \in R$ is a root, then $-\alpha \in R$ is a root as well.*

These facts will all be proved in general in due course; for the time being, they are just things we will observe as we do the analysis of each simple Lie algebra in turn. We mention them here simply because some of what follows will make sense only given these facts. Note in particular that by (ii), the roots all lie in (and span) a real subspace of $\mathfrak{h}^*$; all our pictures clearly will be of this real subspace.

Remark 14.3. If indeed 0 does appear as an eigenvalue of the action of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$, then we may conclude from this that $\mathfrak{h}$ was not maximal to begin with: by the above, anything in the 0-eigenspace of the action of $\mathfrak{h}$ commutes with $\mathfrak{h}$ and (given the fact that the $\mathfrak{g}_\alpha$ are one dimensional) acts diagonally on $\mathfrak{g}$, so that if it not already in $\mathfrak{h}$, then $\mathfrak{h}$ could be enlarged while still retaining the properties of being abelian and diagonalizable. Similarly, the assertion in (ii) that the roots span $\mathfrak{h}^*$ follows from the fact that an element of $\mathfrak{h}$ in the annihilator of all of them would be in the center of $\mathfrak{g}$.

From what we have done so far, we get our first picture of the structure of an arbitrary irreducible finite-dimensional representation V of $\mathfrak{g}$. Specifically, V will admit a direct sum decomposition

$$V = \bigoplus V_\alpha, \tag{14.4}$$

where the direct sum runs over a finite set of $\alpha \in \mathfrak{h}^*$ and $\mathfrak{h}$ acts diagonally on each V_α by multiplication by the eigenvalue α, i.e., for any $H \in \mathfrak{h}$ and $v \in V_\alpha$ we will have

$$H(v) = \alpha(H) \cdot v.$$

The eigenvalues $\alpha \in \mathfrak{h}^*$ that appear in this direct sum decomposition are called the *weights* of V; the V_α themselves are called *weight spaces*; and the dimension of a weight space V_α will be called the *multiplicity* of the weight α in V. We will often represent V by drawing a picture of the set of its weights and thinking of each dot as representing a subspace; this picture (often with some annotation to denote the multiplicity of each weight) is called the *weight diagram* of V.

The action of the rest of the Lie algebra on V can be described in these terms: for any root β, we have

$$\mathfrak{g}_\beta \colon V_\alpha \to V_{\alpha+\beta},$$

so we can think of the action of $\mathfrak{g}_\beta$ on V as a translation in the weight diagram, shifting each of the dots over by β and mapping the weight spaces correspondingly.

Observe next that all the weights of an irreducible representation are congruent to one another modulo the root lattice Λ_R: otherwise, for any weight α of V the subspace

$$V' = \bigoplus_{\beta \in \Lambda_R} V_{\alpha+\beta}$$

would be a proper subrepresentation of V. In particular, in view of Fact 14.2(ii), this means that the weights all lie in a translate of the real subspace spanned by the roots, so that it is not so unreasonable to draw a picture of them.

Step 3. *Find the distinguished subalgebras* $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C} \subset \mathfrak{g}$. As we saw in the example of $\mathfrak{sl}_3\mathbb{C}$, a crucial ingredient in the analysis of an arbitrary irreducible finite-dimensional representation is the restriction of the representation to certain special copies of the algebra $\mathfrak{sl}_2\mathbb{C}$ contained in $\mathfrak{g}$, and the application of what we know from Lecture 11 about such representations. To generalize this to our arbitrary Lie algebra $\mathfrak{g}$, let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be a root space, one dimensional by (i) of Fact 14.2. Then by (iii) of Fact 14.2, there is another root space $\mathfrak{g}_{-\alpha} \subset \mathfrak{g}$; and their commutator $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ must be a subspace of $\mathfrak{g}_0 = \mathfrak{h}$, of dimension at most one. The adjoint action of the commutator $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ thus carries each of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ into itself; so that *the direct sum*

$$\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \tag{14.5}$$

is a subalgebra of $\mathfrak{g}$. The structure of $\mathfrak{s}_\alpha$ is not hard to describe, given two further facts that we will state here, verify in cases, and prove in general in Appendix D.

Facts 14.6.

(i) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$; *and*
(ii) $[[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \mathfrak{g}_\alpha] \neq 0$.

Given these, it follows that *the subalgebra* $\mathfrak{s}_\alpha$ *is isomorphic* to $\mathfrak{sl}_2\mathbb{C}$. In particular, we can pick a basis $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$, and $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ satisfying the standard commutation relations (9.1) for $\mathfrak{sl}_2\mathbb{C}$; X_α and Y_α are not determined by this, but H_α is, being the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ having eigenvalues 2 and -2 on $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, respectively [i.e., H_α is uniquely characterized by the requirements that $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and $\alpha(H_\alpha) = 2$.]

Step 4. *Use the integrality of the eigenvalues of the* H_α. The distinguished elements $H_\alpha \in \mathfrak{h}$ found above are important first of all because, by the analysis of the representations of $\mathfrak{sl}_2\mathbb{C}$ carried out in Lecture 9, in any representation of $\mathfrak{s}_\alpha$—and hence in any representation of $\mathfrak{g}$—*all eigenvalues of the action of* H_α *must be integers*. Thus, every eigenvalue $\beta \in \mathfrak{h}^*$ of every representation of $\mathfrak{g}$ must assume integer values on all the H_α. We correspondingly let Λ_W be the set of linear functionals $\beta \in \mathfrak{h}^*$ that are integer valued on all the H_α; Λ_W will be a lattice, called the *weight lattice* of $\mathfrak{g}$, with the property that

all weights of all representations of $\mathfrak{g}$ *will lie in* Λ_W.

Note, in particular, that $R \subset \Lambda_W$ and hence $\Lambda_R \subset \Lambda_W$; in fact, the root lattice will in general be a sublattice of finite index in the weight lattice.

Step 5. *Use the symmetry of the eigenvalues of the* H_α. The integrality of the eigenvalues of the H_α under any representation is only half the story; it is also true that they are symmetric about the origin in $\mathbb{Z}$. To express this, for any α we introduce the involution W_α on the vector space $\mathfrak{h}^*$ with $+1$-eigenspace the hyperplane

$$\Omega_\alpha = \{\beta \in \mathfrak{h}^*: \langle H_\alpha, \beta\rangle = 0\} \tag{14.7}$$

and minus 1 eigenspace the line spanned by α itself.[1] In English, W_α is the reflection in the plane Ω_α with axis the line spanned by α:

$$W_\alpha(\beta) = \beta - \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)}\alpha = \beta - \beta(H_\alpha)\alpha. \tag{14.8}$$

Let $\mathfrak{W}$ be the group generated by these involutions; $\mathfrak{W}$ is called the *Weyl group* of the Lie algebra $\mathfrak{g}$.

Now suppose that V is any representation of $\mathfrak{g}$, with eigenspace decomposition $V = \bigoplus V_\beta$. The weights β appearing in this decomposition can then be broken up into equivalence classes mod α, and the direct sum

$$V_{[\beta]} = \bigoplus_{n \in \mathbb{Z}} V_{\beta + n\alpha} \tag{14.9}$$

of the eigenspaces in a given equivalence class will be a subrepresentation of V for $\mathfrak{s}_\alpha$. It follows then that the set of weights of V congruent to any given β mod α will be invariant under the involution W_α; in particular,

The set of weights of any representation of $\mathfrak{g}$ *is invariant under the Weyl group.*

To make this more explicit, the string of weights that correspond to nonzero summands in (14.9) are, possibly after replacing β by a translate by a multiple of α:

$$\beta, \beta + \alpha, \beta + 2\alpha, \ldots, \beta + m\alpha, \quad \text{with } m = -\beta(H_\alpha). \tag{14.10}$$

(Note that by our analysis of $\mathfrak{sl}_2\mathbb{C}$ this must be an uninterrupted string.) Indeed if we choose β and $m \geq 0$ so that (14.10) is the string corresponding to nonzero summands in (14.9), then the string of integers

$$\beta(H_\alpha), (\beta + \alpha)(H_\alpha) = \beta(H_\alpha) + 2, \ldots, (\beta + m\alpha)(H_\alpha) = \beta(H_\alpha) + 2m$$

must be symmetric about zero, so $\beta(H_\alpha) = -m$. In particular,

$$W_\alpha(\beta + k\alpha) = \beta + (-\beta(H_\alpha) - k)\alpha = \beta + (m - k)\alpha.$$

Note also that by the same analysis the multiplicities of the weights are invariant under the Weyl group.

We should mention one other fact about the Weyl group, whose proof we also postpone:

[1] Note that by the nondegeneracy assertion (ii) of Fact 14.6, the line $\mathbb{C}\cdot\alpha$ does not lie in the hyperplane Ω_α. Recall that $\langle\ ,\ \rangle$ is the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$, so $\langle H_\alpha, \beta\rangle = \beta(H_\alpha)$.

Fact 14.11. *Every element of the Weyl group is induced by an automorphism of the Lie algebra* $\mathfrak{g}$ *carrying* $\mathfrak{h}$ *to itself.*

We can even say what automorphism of $\mathfrak{g}$ does the trick: to get the involution W_α, take the adjoint action of the exponential $\exp(\pi i U_\alpha) \in G$, where G is any group with Lie algebra $\mathfrak{g}$ and U_α is a suitable element of the direct sum of the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. To prove that $\mathrm{Ad}(\exp(\pi i U_\alpha))$ actually does this requires more knowledge of $\mathfrak{g}$ than we currently possess; but it would be an excellent exercise to verify this assertion directly in each of the cases studied below. (For the general case see (23.20) and (26.15).)

Step 6. *Draw the picture* (optional). While there is no logical need to do so at this point, it will be much easier to think about what is going on in $\mathfrak{h}^*$ if we introduce the appropriate inner product, called the *Killing form*, on $\mathfrak{g}$ (hence by restriction on $\mathfrak{h}$, and hence on $\mathfrak{h}^*$). Since the introduction of the Killing form is, logically, a digression, we will defer until later in this lecture a discussion of its various definitions and properties. It will suffice for now to mention the characteristic property of the induced inner product on $\mathfrak{h}^*$: up to scalars it is the unique inner product on $\mathfrak{h}^*$ preserved by the Weyl group, i.e., in terms of which the Weyl group acts as a group of orthogonal transformations. Equivalently, it is the unique inner product (up to scalars) such that the line spanned by each root $\alpha \in \mathfrak{h}^*$ is actually perpendicular to the plane Ω_α (so that the involution W_α is just a reflection in that hyperplane). Indeed, in practice this is most often how we will compute it. In terms of the Killing form, then, we can say that *the Weyl group is just the group generated by the reflections in the hyperplanes perpendicular to the roots of the Lie algebra.*

Step 7. *Choose a direction in* $\mathfrak{h}^*$. By this we mean a real linear functional l on the lattice Λ_R irrational with respect to this lattice. This gives us a decomposition of the set

$$R = R^+ \cup R^-, \tag{14.12}$$

where $R^+ = \{\alpha\colon l(\alpha) > 0\}$ (the $\alpha \in R^+$ are called the *positive* roots, those in R^- *negative*); this decomposition is called an *ordering of the roots.* For most purposes, the only aspect of l that matters is the associated ordering of the roots.

The point of choosing a direction—and thereby an ordering of the roots $R = R^+ \cup R^-$—is, of course, to mimic the notion of highest weight vector that was so crucial in the cases of $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$. Specifically, we make the

Definition. Let V be any representation of $\mathfrak{g}$. A nonzero vector $v \in V$ that is both an eigenvector for the action of $\mathfrak{h}$ and in the kernel of $\mathfrak{g}_\alpha$ for all $\alpha \in R^+$ is called a *highest weight vector* of V.

Just as in the previous cases, we then have

Proposition 14.13. *For any semisimple complex Lie algebra* $\mathfrak{g}$,

(i) *every finite-dimensional representation* V *of* $\mathfrak{g}$ *possesses a highest weight vector*;

(ii) *the subspace W of V generated by the images of a highest weight vector v under successive applications of root spaces $\mathfrak{g}_\beta$ for $\beta \in R^-$ is an irreducible subrepresentation;*

(iii) *an irreducible representation possesses a unique highest weight vector up to scalars.*

PROOF. Part (i) is immediate: we just take α to be the weight appearing in V for which the value $l(\alpha)$ is maximal and choose v any nonzero vector in the weight space V_α. Since $V_{\alpha+\beta} = (0)$ for all $\beta \in R^+$, such a vector v will necessarily be in the kernel of all root spaces $\mathfrak{g}_\beta$ corresponding to positive roots β.

Part (ii) may be proved by the same argument as in the two cases we have already discussed: we let W_n be the subspace spanned by all $w_n \cdot v$ where w_n is a word of length at most n in elements of $\mathfrak{g}_\beta$ for negative β. We then claim that for any X in any positive root space, $X \cdot W_n \subset W_n$. To see this, write a generator of W_n in the form $Y \cdot w$, $w \in W_{n-1}$, and use the commutation relation $X \cdot Y \cdot w = Y \cdot X \cdot w + [X, Y] \cdot w$; the claim follows by induction, since $[X, Y]$ is always in $\mathfrak{h}$. The subspace $W \subset V$ which is a union of all the W_n's is thus a subrepresentation; to see that it is irreducible; note that if we write $W = W' \oplus W''$, then either W' or W'' will have to contain the one-dimensional weight space W_α, and so will have to equal W.

The uniqueness of the highest weight vector of an irreducible representation follows immediately: if $v \in V_\alpha$ and $w \in V_\beta$ were two such, not scalar multiples of each other, we would have $l(\alpha) > l(\beta)$ and vice versa. □

Exercise 14.14. Show that in (ii) one need only apply those $\mathfrak{g}_\beta$ for which $\mathfrak{g}_\beta \cdot v \neq 0$. (Note: with W_n defined using only these $\mathfrak{g}_\beta$, and X in any root space, the same inductive argument shows that $X \cdot W_n \subset W_{n+1}$. On the other hand, if one uses all $\mathfrak{g}_\beta$ with β negative and primitive, as in Observation 14.16, then $X \cdot W_n \subset W_{n-1}$. One cannot combine these, however: V may *not* be generated by successively applying those $\mathfrak{g}_\beta$ with β negative, primitive, and $\mathfrak{g}_\beta \cdot v \neq 0$, e.g., the standard representation of $\mathfrak{sl}_3\mathbb{C}$.)

The weight α of the highest weight vector of an irreducible representation will be called, not unreasonably, the *highest weight* of that representation; the term *dominant weight* is also common.

We can refine part (ii) of this proposition slightly in another direction; this is not crucial but will be useful later on in estimating multiplicities of various representations. This refinement is based on

Exercise 14.15*. (a) Let $\alpha_1, \ldots, \alpha_k$ be roots of a semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{g}_{\alpha_i} \subset \mathfrak{g}$ the corresponding root spaces. Show that the subalgebra of $\mathfrak{g}$ generated by the Cartan subalgebra $\mathfrak{h}$ together with the $\mathfrak{g}_{\alpha_i}$ is exactly the direct sum $\mathfrak{h} \oplus (\bigoplus \mathfrak{g}_\alpha)$, where the direct sum is over the intersection of the set R of roots of $\mathfrak{g}$ with the semigroup $\mathbb{N}\{\alpha_1, \ldots, \alpha_k\} \subset \mathfrak{h}$ generated by the α_i.

(b) Similarly, let $\alpha_1, \ldots, \alpha_k$ be negative roots of a semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{g}_{\alpha_i} \subset \mathfrak{g}$ the corresponding root spaces. Show that the subalgebra of $\mathfrak{g}$ generated by the $\mathfrak{g}_{\alpha_i}$ is exactly the direct sum $\bigoplus \mathfrak{g}_\alpha$, where the direct sum is over the intersection of the set R of roots of $\mathfrak{g}$ with the semigroup $\mathbb{N}\{\alpha_1, \ldots, \alpha_k\} \subset \mathfrak{h}$ generated by the α_i.

(Note that by the description of the adjoint action of a Lie algebra on itself we have an obvious inclusion; the problem here is to show—given the facts above—that if $\alpha + \beta \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$.)

From this exercise, it is clear that generating a subrepresentation W of a given representation V by successive applications of root spaces $\mathfrak{g}_\beta$ for $\beta \in R^-$ to a highest weight vector v is inefficient; we need only apply the root spaces $\mathfrak{g}_\beta$ corresponding to a set of roots β generating R^- as a semigroup. We accordingly introduce another piece of terminology: we say that a positive (resp., negative) root $\alpha \in R$ is *primitive* or *simple* if it cannot be expressed as a sum of two positive (resp. negative) roots. (Note that, since there are only finitely many roots, every positive root can be written as a sum of primitive positive roots.) We then have

Observation 14.16. *Any irreducible representation V is generated by the images of its highest weight vector v under successive applications of root spaces $\mathfrak{g}_\beta$ where β ranges over the primitive negative roots.*

We have already seen one example of this in the case of $\mathfrak{sl}_3\mathbb{C}$, where we observed (in the proof of Claim 12.10 and in the analysis of $\mathrm{Sym}^2 V \otimes V^*$ in Lecture 13) that any irreducible representation was generated by applying the two elements $E_{2,1} \in \mathfrak{g}_{L_2 - L_1}$ and $E_{3,2} \in \mathfrak{g}_{L_3 - L_2}$ to a highest weight vector.

To return to our description of the weights of an irreducible representation V, we observe next that in fact *every vertex of the convex hull of the weights of V must be conjugate to α under the Weyl group.* To see this, note that by the above the set of weights is contained in the cone $\alpha + C_\alpha^-$, where C_α^- is the positive real cone spanned by the roots $\beta \in R^-$ such that $\mathfrak{g}_\beta(v) \neq 0$—that is, such that $\alpha(H_\beta) \neq 0$. Conversely, the weights of V will contain the string of weights

$$\alpha, \alpha + \beta, \alpha + 2\beta, \ldots, \alpha + (-\alpha(H_\beta))\beta \tag{14.17}$$

for any $\beta \in R^-$. Thus, any vertex of the convex hull of the set of weights of V adjacent to α must be of the form

$$\alpha - \alpha(H_\beta)\beta = W_\beta(\alpha)$$

for some β; applying the same analysis to each successive vertex gives the statement.

From the above, we deduce that the set of weights of V will lie in the convex hull of the images of α under the Weyl group. Since, moreover, we know that the intersection of this set with any set of weights of the form $\{\beta + n\gamma\}$ will be a connected string, it follows that *the set of weights of V will be exactly the weights that are congruent to α modulo the root lattice Λ_R and that lie in the convex hull of the images of α under the Weyl group.*

One more bit of terminology, and then we are done. By what we have seen (cf. (14.17)), the highest weight of any representation of V will be a weight α satisfying $\alpha(H_\gamma) \geq 0$ for every $\gamma \in R^+$. The locus $\mathscr{W}$, in the real span of the roots, of points satisfying these inequalities—in terms of the Killing form, making an acute or right angle with each of the positive roots—is called the (closed) *Weyl chamber* associated to the ordering of the roots. A Weyl chamber could also be described as the closure of a connected component of the complement of the union of the hyperplanes Ω_α. The Weyl group acts simply transitively on the set of Weyl chambers and likewise on the set of orderings of the roots. As usual, these statements will be easy to see in the cases we study, while the abstract proofs are postponed (to Appendix D).

Step 8. *Classify the irreducible, finite-dimensional representations of* $\mathfrak{g}$. Where all the above is leading should be pretty clear; it is expressed in the fundamental existence and uniqueness theorem:

Theorem 14.18. *For any α in the intersection of the Weyl chamber $\mathscr{W}$ associated to the ordering of the roots with the weight lattice Λ_W, there exists a unique irreducible, finite-dimensional representation Γ_α of $\mathfrak{g}$ with highest weight α; this gives a bijection between $\mathscr{W} \cap \Lambda_W$ and the set of irreducible representations of $\mathfrak{g}$. The weights of Γ_α will consist of those elements of the weight lattice congruent to α modulo the root lattice Λ_R and lying in the convex hull of the set of points in $\mathfrak{h}^*$ conjugate to α under the Weyl group.*

HALF-PROOF. We will give here just the proof of uniqueness, which is easy. The existence part we will demonstrate explicitly in each example in turn; and later on we will sketch some of the constructions that can be made in general.

The uniqueness part is exactly the same as for $\mathfrak{sl}_3\mathbb{C}$. If V and W are two irreducible, finite-dimensional representations of $\mathfrak{g}$ with highest weight vectors v and w, respectively, both having weight α, then the vector $(v, w) \in V \oplus W$ will again be a highest weight vector of weight α in that representation. Let $U \subset V \oplus W$ be the subrepresentation generated by (v, w); since U will again be irreducible the projection maps $\pi_1: U \to V$ and $\pi_2: U \to W$, being nonzero, will have to be isomorphisms. □

Another fact which we will see as we go along—and eventually prove in general—is that there are always *fundamental weights* $\omega_1, \ldots, \omega_n$ with the property that any dominant weight can be expressed uniquely as a non-negative integral linear combination of them. They can be characterized geometrically as the first weights met along the edges of the Weyl chamber, or algebraically as those elements ω_i in $\mathfrak{h}^*$ such that $\omega_i(H_{\alpha_j}) = \delta_{i,j}$, where $\alpha_1, \ldots, \alpha_n$ are the simple roots (in some order). When we have found them, we often write $\Gamma_{a_1,\ldots,a_n}$ for the irreducible representation with highest weight $a_1\omega_1 + \cdots + a_n\omega_n$; i.e.,

$$\Gamma_{a_1,\ldots,a_n} = \Gamma_{a_1\omega_1 + \cdots + a_n\omega_n}.$$

As with most of the material in this section, general proofs will be found in Lecture 21 and Appendix D.

One basic point we want to repeat here (and that we hope to demonstrate in succeeding lectures) is this: that actually carrying out this process in practice is completely elementary and straightforward. Any mathematician, stranded on a desert island with only these ideas and the definition of a particular Lie algebra $\mathfrak{g}$ such as $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{so}_n\mathbb{C}$, or $\mathfrak{sp}_{2n}\mathbb{C}$, would in short order have a complete description of all the objects defined above in the case of $\mathfrak{g}$. We should say as well, however, that at the conclusion of this procedure we are left without one vital piece of information about the representations of $\mathfrak{g}$, without which we will be unable to analyze completely, for example, tensor products of known representations; this is, of course, a description of the multiplicities of the basic representations Γ_α. As we said, we will, in fact, describe and prove such a formula (the Weyl character formula); but it is of a much less straightforward character (our hypothetical shipwrecked mathematician would have to have what could only be described as a pretty good day to come up with the idea) and will be left until later. For now, we will conclude this lecture with the promised introduction to the Killing form.

§14.2. About the Killing Form

As we said, the Killing form is an inner product (symmetric bilinear form) on the Lie algebra $\mathfrak{g}$; abusing our notation, we will denote by B both the Killing form and the induced inner products on $\mathfrak{h}$ and $\mathfrak{h}^*$. B can be defined in several ways; the most common is by associating to a pair of elements $X, Y \in \mathfrak{g}$ the trace of the composition of their adjoint actions on $\mathfrak{g}$, i.e.,

$$B(X, Y) = \operatorname{Tr}(\operatorname{ad}(X) \circ \operatorname{ad}(Y)\colon \mathfrak{g} \to \mathfrak{g}). \tag{14.19}$$

As we will see, the Killing form may be computed in practice either from this definition, or (up to scalars) by using its invariance under the group of automorphisms of $\mathfrak{g}$. We remark that this definition is not as opaque as it may seem at first. For one thing, the description of the adjoint action of the root space $\mathfrak{g}_\alpha$ as a "translation" of the root diagram—that is, carrying each root space $\mathfrak{g}_\beta$ into $\mathfrak{g}_{\alpha+\beta}$—tells us immediately that $\mathfrak{g}_\alpha$ is perpendicular to $\mathfrak{g}_\beta$ for all β other than $-\alpha$; in other words, the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \right) \tag{14.20}$$

is orthogonal. As for the restriction of B to $\mathfrak{h}$, this is more subtle, but it is not hard to write down: if X, Y are in $\mathfrak{h}$, and Z_α generates $\mathfrak{g}_\alpha$, then $\operatorname{ad}(X) \circ \operatorname{ad}(Y)(Z_\alpha) = \alpha(X)\alpha(Y)Z_\alpha$, so $B(X, Y) = \sum \alpha(X)\alpha(Y)$, the sum over the roots; viewing $B|_{\mathfrak{h}}$ as an element of the symmetric square $\operatorname{Sym}^2(\mathfrak{h}^*)$, we have

$$B|_{\mathfrak{h}} = \frac{1}{2} \sum_{\alpha \in R} \alpha^2. \tag{14.21}$$

A key fact following from this—one that, if nothing else, makes picturing $\mathfrak{h}^*$ with the inner product B involve less eyestrain—is

(14.22) *B is positive definite on the real subspace of $\mathfrak{h}$ spanned by the vectors $\{H_\alpha : \alpha \in R\}$.*

Indeed, all roots take on real values on this space (since all $\alpha(H_\beta) \in \mathbb{Z} \subset \mathbb{R}$), so for H in this real subspace of $\mathfrak{h}$, $B(H, H)$ is non-negative, and is zero only when all $\alpha(H) = 0$, which implies $H = 0$, since the roots span $\mathfrak{h}^*$.

To see that the Killing form is nondegenerate on all of $\mathfrak{g}$, we need the useful identity:

$$B([X, Y], Z) = B(X, [Y, Z]) \tag{14.23}$$

for all X, Y, Z in $\mathfrak{g}$. This follows from the identity

$$\mathrm{Trace}((\bar{X}\bar{Y} - \bar{Y}\bar{X})\bar{Z}) = \mathrm{Trace}(\bar{X}(\bar{Y}\bar{Z} - \bar{Z}\bar{Y}))$$

for any endomorphisms $\bar{X}$, $\bar{Y}$, $\bar{Z}$ of a vector space. And this, in turn, follows from

$$\mathrm{Trace}(\bar{Y}\bar{X}\bar{Z} - \bar{X}\bar{Z}\bar{Y}) = \mathrm{Trace}([\bar{Y}, \bar{X}\bar{Z}]) = 0.$$

An immediate consequence of (14.23) is that if $\mathfrak{a}$ is any ideal in a Lie algebra $\mathfrak{g}$, then its orthogonal complement $\mathfrak{a}^\perp$ with respect to B is also an ideal. In particular, if $\mathfrak{g}$ is simple, the kernel of B is zero (note that the kernel cannot be $\mathfrak{g}$ since it does not contain $\mathfrak{h}$). Since the Killing form of a direct sum is the sum of the Killing forms of the factors, it follows that *the Killing form is nondegenerate on a semisimple Lie algebra* $\mathfrak{g}$.

One of the reasons the Killing form helps to picture $\mathfrak{h}^*$ is the fact mentioned above:

Proposition 14.24. *With respect to B, the line spanned by each root α is perpendicular to the hyperplane Ω_α.*

As we observed, this is equivalent to saying that the involutions W_α above are simply reflections in hyperplanes, and in turn to saying that the whole Weyl group is orthogonal. Note also that Proposition 14.24 thereby follows immediately from the Fact 14.11: from the definition of B above, it is clearly invariant under any automorphism of $\mathfrak{g}$. Nevertheless, we would prefer not to rely on this fact; and anyway giving a direct proof of the proposition is not hard, in terms of the picture we have of the adjoint action of $\mathfrak{g}$ on itself. To prove the assertion $\alpha \perp \Omega_\alpha$, it suffices to prove the dual assertion that $H \perp H_\alpha$ for all H in the annihilator of α. But now by construction H_α is the commutator $[X_\alpha, Y_\alpha]$ of an element $X_\alpha \in \mathfrak{g}_\alpha$ and an element $Y_\alpha \in \mathfrak{g}_{-\alpha}$. Using (14.23) we have for any H in $\mathfrak{h}$,

$$\begin{aligned} B(H_\alpha, H) &= B([X_\alpha, Y_\alpha], H) = B(X_\alpha, [Y_\alpha, H]) \\ &= B(X_\alpha, \alpha(H) Y_\alpha) = \alpha(H) B(X_\alpha, Y_\alpha), \end{aligned} \tag{14.25}$$

which vanishes since $\alpha(H) = 0$.

Note that as a consequence of this, we can characterize the Weyl chamber associated to an ordering of the roots as exactly those vectors in the real span of the roots forming an acute angle with all the positive roots (or, equivalently, with all the primitive ones); the Weyl chamber is thus the cone whose faces lie in the hyperplanes perpendicular to the primitive positive roots.

Equation (14.25) leads to a formula for the isomorphism of $\mathfrak{h}$ with $\mathfrak{h}^*$ determined by the Killing form. First note that for $H = H_\alpha$ it gives

$$B(H_\alpha, H_\alpha) = 2B(X_\alpha, Y_\alpha) \neq 0,$$

for if $B(X_\alpha, Y_\alpha)$ were zero we would have $B(H_\alpha, H) = 0$ for all H, contradicting the nondegeneracy of B on $\mathfrak{h}$. The element T_α of $\mathfrak{h}$ which corresponds to $\alpha \in \mathfrak{h}^*$ by the Killing form is by definition the element of $\mathfrak{h}$ that satisfies the condition

$$B(T_\alpha, H) = \alpha(H) \quad \text{for all } H \in \mathfrak{h}. \tag{14.26}$$

Looking at (14.25), we see that $T_\alpha = H_\alpha/B(X_\alpha, Y_\alpha) = 2H_\alpha/B(H_\alpha, H_\alpha)$. This proves

Corollary 14.27. *The isomorphism of $\mathfrak{h}^*$ and $\mathfrak{h}$ determined by the Killing form B carries α to $T_\alpha = (2/B(H_\alpha, H_\alpha))\cdot H_\alpha$.*

The Killing form on $\mathfrak{h}^*$ is defined by $B(\alpha, \beta) = B(T_\alpha, T_\beta)$.

Exercise 14.28. Show that the inverse isomorphism from $\mathfrak{h}$ to $\mathfrak{h}^*$ takes H_α to $(2/B(\alpha, \alpha))\cdot\alpha$.

The orthogonality of W_α can be expressed by the formula

$$W_\alpha(\beta) = \beta - \frac{2B(\beta, \alpha)}{B(\alpha, \alpha)}\alpha.$$

Comparing with (14.8) this says:

Corollary 14.29. *If α and β are roots, then*

$$2B(\beta, \alpha)/B(\alpha, \alpha) = \beta(H_\alpha)$$

is an integer.

By the above identification of $\mathfrak{h}$ with $\mathfrak{h}^*$, (14.22) translates to

Corollary 14.30. *The Killing form B is positive definite on the real vector space spanned by the root lattice Λ_R.*

Note that it follows immediately from (14.22) that the Weyl group $\mathfrak{W}$ is finite, being simultaneously discrete ($\mathfrak{W}$ preserves the set R of roots of $\mathfrak{g}$ and hence the lattice Λ_R; it follows that $\mathfrak{W}$ can be realized as a subgroup of $\mathrm{GL}_n\mathbb{Z}$)

and compact ($\mathfrak{W}$ preserves the Killing form, and hence is a subgroup of the orthogonal group $O_n\mathbb{R}$.) Alternatively, $\mathfrak{W}$ is a subgroup of the permutation group of the set of roots.

As we observed, the Killing form on $\mathfrak{h}^*$ is preserved by the Weyl group. In fact, in case $\mathfrak{g}$ is simple, the Killing form is, up to scalars, the unique inner product preserved by the Weyl group. This will follow from

Proposition 14.31. *The space $\mathfrak{h}^*$ is an irreducible representation of the Weyl group $\mathfrak{W}$.*

Proof. Suppose that $\mathfrak{z} \subset \mathfrak{h}^*$ were preserved by the action of $\mathfrak{W}$. This means that every root $\alpha \in \mathfrak{h}^*$ of $\mathfrak{g}$ will either lie in the subspace $\mathfrak{z}$ or be perpendicular to it, i.e., for every $\alpha \in \mathfrak{z}$ and $\beta \notin \mathfrak{z}$ we will have $\beta(H_\alpha) = 0$. We claim then that *the subspace $\mathfrak{g}'$ of $\mathfrak{g}$ spanned by the subalgebras $\{\mathfrak{s}_\alpha\}_{\alpha \in \mathfrak{z}}$ will be an ideal in $\mathfrak{g}$.* Clearly it will be a subalgebra; the space spanned by the distinguished subalgebras $\mathfrak{s}_\alpha$ corresponding to the set of roots lying in any subspace of $\mathfrak{h}^*$ will be. To see that it is in fact an ideal, let $Y \in \mathfrak{g}_\beta$ be an element of a root space. Then for any $\alpha \in \mathfrak{z}$, we have

$$[Y, Z] \in \mathfrak{g}_{\alpha+\beta} = 0$$

since $\alpha + \beta$ is neither in $\mathfrak{z}$ nor perpendicular to it, and so cannot be a root; and

$$[Y, H_\alpha] = -[H_\alpha, Y] = \beta(H_\alpha) \cdot Y = 0.$$

Thus, $\operatorname{ad}(Y)$ kills $\mathfrak{g}'$; since, of course, all of H itself will preserve $\mathfrak{g}'$, it follows that $\mathfrak{g}'$ is an ideal. Thus, either all the roots lie in $\mathfrak{z}$ and so $\mathfrak{z} = \mathfrak{h}^*$, or all roots are perpendicular to $\mathfrak{z}$ and correspondingly $\mathfrak{z} = (0)$. □

Note that given Fact 14.11, we can also express the last statement by saying that (in case $\mathfrak{g}$ is simple) the Killing form on $\mathfrak{h}$ is the unique form preserved by every automorphism of the Lie algebra $\mathfrak{g}$ carrying $\mathfrak{h}$ to itself. As we will see, in practice this is most often how we will first describe the Killing form.

Exercise 14.32. Find the Killing form on the Lie algebras $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$ by explicit computation, and verify the statements made above in these cases.

Exercise 14.33*. If a semisimple Lie algebra is a direct sum of simple subalgebras, then its Killing form is the orthogonal sum of the Killing forms of the factors. Show that, conversely, if the roots of a semisimple Lie algebra lie in a collection of mutually perpendicular subspaces, then the Lie algebra decomposes accordingly.

Exercise 14.34*. Suppose $\mathfrak{g}$ is a Lie algebra that has an abelian subalgebra $\mathfrak{h}$ such that $\mathfrak{g}$ has a decomposition (14.1), satisfying the conditions of Facts 14.2 and 14.6. Show that $\mathfrak{g}$ is semisimple, and $\mathfrak{h}$ is a Cartan subalgebra.

The preceding exercise can be used instead of Weyl's unitary trick or any abstract theory to verify that the algebras we meet in the next few lectures are all semisimple. It is tempting to call such a Lie algebra "visibly semisimple."

The discussion of the geometry of the roots of a semisimple Lie algebra will be continued in Lecture 21 and completed in Appendix D. The Killing form becomes particularly useful in the general theory; for example, solvability and semisimplicity can both be characterized by properties of the Killing form (see Appendix C).

Exercise 14.35*. Show that $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$ is a maximal solvable subalgebra of $\mathfrak{g}$; $\mathfrak{b}$ is called a *Borel subalgebra.* Show that $\bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$ is a maximal nilpotent subalgebra of $\mathfrak{g}$. These will be discussed in Lecture 25.

Exercise 14.36*. Show that the Killing form on the Lie algebra $\mathfrak{gl}_m$ is given by the formula

$$B(X, Y) = 2m \operatorname{Tr}(X \circ Y) - 2 \operatorname{Tr}(X) \operatorname{Tr}(Y).$$

Find similar formulas for $\mathfrak{sl}_m$, $\mathfrak{so}_m$, and $\mathfrak{sp}_m$, showing in each case that $B(X, Y)$ is a constant multiple of $\operatorname{Tr}(X \circ Y)$.

Exercise 14.37. If G is a real Lie group, the Killing form on its Lie algebra $\mathfrak{g} = T_e G$ may not be positive definite. When it is, it determines, by left translation, a Riemannian metric on G. Show that the Killing form is positive definite for $G = \mathrm{SO}_n\mathbb{R}$, but not for $\mathrm{SL}_n\mathbb{R}$.

LECTURE 15
$\mathfrak{sl}_4\mathbb{C}$ and $\mathfrak{sl}_n\mathbb{C}$

In this lecture, we will illustrate the general paradigm of the previous lecture by applying it to the Lie algebras $\mathfrak{sl}_n\mathbb{C}$; this is typical of the analyses of specific Lie algebras carried out in this Part. We start in §15.1 by describing the Cartan subalgebra, roots, root spaces, etc., for $\mathfrak{sl}_n\mathbb{C}$ in general. We then give in §15.2 a detailed account of the representations of $\mathfrak{sl}_4\mathbb{C}$, which generalizes directly to $\mathfrak{sl}_n\mathbb{C}$; in particular, we deduce the existence part of Theorem 14.18 for $\mathfrak{sl}_n\mathbb{C}$.

In §15.3 we give an explicit construction of the irreducible representations of $\mathfrak{sl}_n\mathbb{C}$ using the Weyl construction introduced in Lecture 6; analogous constructions of the irreducible representations of the remaining classical Lie algebras will be given in §17.3 and §19.5. This section presupposes familiarity with Lecture 6 and Appendix A, but can be skipped by those willing to forego §17.3 and 19.5 as well. Section 15.4 requires essentially the same degree of knowledge of classical algebraic geometry as §§11.3 and 13.4 (it does not presuppose §15.3), but can also be skipped. Finally, §15.5 describes representations of $GL_n\mathbb{C}$; this appears to involve the Weyl construction but in fact the main statement, Proposition 15.47 (and even its proof) can be understood without the preceding two sections.

§15.1. Analyzing $\mathfrak{sl}_n\mathbb{C}$

To begin with, we have to locate a Cartan subalgebra, and this is not hard; as in the case of $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$ the subalgebra of diagonal matrices will work fine. Writing H_i for the diagonal matrix $E_{i,i}$ that takes e_i to itself and kills e_j

for $j \neq i$, we have

$$\mathfrak{h} = \{a_1 H_1 + a_2 H_2 + \cdots + a_n H_n : a_1 + a_2 + \cdots + a_n = 0\};$$

note that H_i is not in $\mathfrak{h}$. We can correspondingly write

$$\mathfrak{h}^* = \mathbb{C}\{L_1, L_2, \ldots, L_n\}/(L_1 + L_2 + \cdots + L_n = 0),$$

where $L_i(H_j) = \delta_{i,j}$. We often write L_i for the image of L_i in $\mathfrak{h}^*$.

We have already seen how the diagonal matrices act on the space of all traceless matrices: if $E_{i,j}$ is the endomorphism of $\mathbb{C}^n$ carrying e_j to e_i and killing e_k for all $k \neq j$, then we have

$$\mathrm{ad}(a_1 H_1 + a_2 H_2 + \cdots + a_n H_n)(E_{i,j}) = (a_i - a_j)\cdot E_{i,j}; \tag{15.1}$$

or, in other words, *$E_{i,j}$ is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $L_i - L_j$*; in particular, *the roots of $\mathfrak{sl}_n\mathbb{C}$ are just the pairwise differences of the L_i*.

Before we try to visualize anything taking place in $\mathfrak{h}$ or $\mathfrak{h}^*$, let us take a moment out and describe the Killing form. To this end, note that the automorphism φ of $\mathbb{C}^n$ sending e_i to e_j, e_j to $-e_i$ and fixing e_k for all $k \neq i, j$ induces an automorphism $\mathrm{Ad}(\varphi)$ of the Lie algebra $\mathfrak{sl}_n\mathbb{C}$ (or even $\mathfrak{gl}_n(\mathbb{C})$) that carries $\mathfrak{h}$ to itself, exchanges H_i and H_j, and fixes all the other H_k. Since the Killing form on $\mathfrak{h}$ must be invariant under all these automorphisms, it must satisfy $B(L_i, L_i) = B(L_j, L_j)$ for all i and j and $B(L_i, L_k) = B(L_j, L_k)$ for all i, j and $k \neq i, j$; it follows that on $\mathfrak{h}$ it must be a linear combination of the forms

$$B'(\textstyle\sum a_i H_i, \sum b_i H_i) = \sum a_i b_i$$

and

$$B''(\textstyle\sum a_i H_i, \sum b_i H_i) = \sum_{i \neq j} a_i b_j.$$

On the space $\{\sum a_i H_i : \sum a_i = 0\}$, however, we have $0 = (\sum a_i)(\sum b_j) = \sum a_i b_i + \sum a_i b_j$, so in fact these two forms are dependent; and hence we can write the Killing form simply as a multiple of B'. Similarly, the Killing form on $\mathfrak{h}^*$ must be a linear combination of the forms $B'(\sum a_i L_i, \sum b_i L_i) = \sum a_i b_i$ and $B''(\sum a_i L_i, \sum b_i L_i) = \sum_{j \neq i} a_i b_j$; the condition that $B(\sum a_i L_i, \sum b_i L_i) = 0$ whenever $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$ implies that it must be a multiple of

$$\begin{aligned} B(\textstyle\sum a_i L_i, \sum b_i L_i) &= \left(\frac{n-1}{n}\right)\sum_i a_i b_i - \frac{1}{n}\sum_{i \neq j} a_i b_j \\ &= \sum_i a_i b_i - \frac{1}{n}\sum_{i,j} a_i b_j. \end{aligned} \tag{15.2}$$

We may, of course, also calculate the Killing form directly from the definition. By (14.21), since the roots of $\mathfrak{sl}_n\mathbb{C}$ are $\{L_i - L_j\}_{i \neq j}$, we have

$$\begin{aligned} B(\textstyle\sum a_i H_i, \sum b_i H_i) &= \textstyle\sum_{i \neq j} (a_i - a_j)(b_i - b_j) \\ &= \textstyle\sum_i \sum_{j \neq i} (a_i b_i + a_j b_j - a_i b_j - a_j b_i). \end{aligned}$$

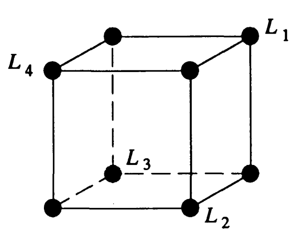
Noting that $\sum_{j \neq i} a_j = -a_i$ and, similarly, $\sum_{j \neq i} b_j = -b_i$, this simplifies to

$$B(\sum a_iH_i, \sum b_iH_i) = 2n \sum a_ib_i. \tag{15.3}$$

It follows with a little calculation that the dual form on $\mathfrak{h}^*$ is

$$B(\sum a_iL_i, \sum b_iL_i) = (1/2n)(\textstyle\sum_i a_ib_i - (1/n) \sum_{i,j} a_ib_j). \tag{15.4}$$

It is probably simpler just to think of this as the form, unique up to scalars, invariant under the symmetric group $\mathfrak{S}_n$ of permutations of $\{1, 2, \ldots, n\}$. The L_i, therefore, all have the same length, and the angles between all pairs are the same. To picture the roots in $\mathfrak{h}^*$, then, we should think of the points L_i as situated at the vertices of a regular $(n - 1)$-simplex Δ, with the origin located at the barycenter of that simplex. This picture is easiest to visualize in the special case $n = 4$, where the L_i will be located at every other vertex of a unit cube centered at the origin:

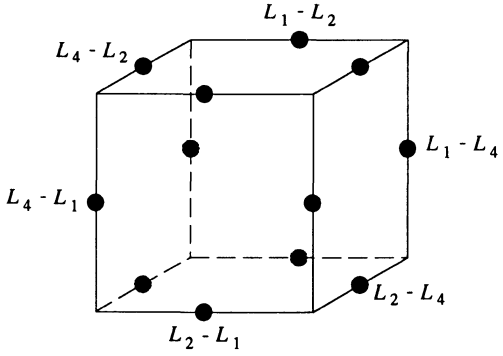


(15.5)

Now, as we said, the roots of $\mathfrak{sl}_n\mathbb{C}$ are now just the pairwise differences of the L_i. The root lattice Λ_R they generate can thus be described as

$$\Lambda_R = \{\sum a_iL_i\colon a_i \in \mathbb{Z}, \sum a_i = 0\}/(\sum L_i = 0).$$

Both the roots and the root lattice can be drawn in the case of $\mathfrak{sl}_4\mathbb{C}$: if we think of the vectors $L_i \in \mathfrak{h}^*$ as four of the vertices of a cube centered at the origin, the roots will comprise all the midpoints of the edges of a second cube whose linear dimensions are twice the dimensions of the first:



(15.6)

The next step, finding the distinguished subalgebras $\mathfrak{s}_\alpha$, is also very easy. The root space $\mathfrak{g}_{L_i-L_j}$ corresponding to the root $L_i - L_j$ is generated by $E_{i,j}$, so the subalgebra $\mathfrak{s}_{L_i-L_j}$ is generated by

$$E_{i,j}, \quad E_{j,i}, \quad \text{and} \quad [E_{i,j}, E_{j,i}] = H_i - H_j.$$

The eigenvalue of $H_i - H_j$ acting on $E_{i,j}$ is $(L_i - L_j)(H_i - H_j) = 2$, so that the corresponding distinguished element $H_{L_i-L_j}$ in $\mathfrak{h}$ must be just $H_i - H_j$. The annihilator, of course, is the hyperplane $\Omega_{L_i-L_j} = \{\sum a_i L_i\colon a_i = a_j\}$; note that this is indeed perpendicular to the root $L_i - L_j$ with respect to the Killing form B as described above.

Knowing the H_α we know the weight lattice: in order for a linear functional $\sum a_i L_i \in \mathfrak{h}^*$ to have integral values on all the distinguished elements, it is clearly necessary and sufficient that all the a_i be congruent to one another modulo $\mathbb{Z}$. Since $\sum L_i = 0$ in $\mathfrak{h}^*$, this means that the weight lattice is given as

$$\Lambda_W = \mathbb{Z}\{L_1, \ldots, L_n\}/(\textstyle\sum L_i = 0).$$

In sum, then, *the weight lattice of* $\mathfrak{sl}_n\mathbb{C}$ *may be realized as the lattice generated by the vertices of a regular* $(n-1)$*-simplex* Δ *centered at the origin*; *and the roots as the pairwise differences of these vertices.*

While we are at it, having determined Λ_R and Λ_W we might as well compute the quotient Λ_W/Λ_R. This is pretty easy: since the lattice Λ_W can be generated by Λ_R together with any of the vertices L_i of our simplex, the quotient Λ_W/Λ_R will be cyclic, generated by any L_i; since, modulo Λ_R,

$$0 = \textstyle\sum_j (L_i - L_j) = nL_i - \sum_j L_j = nL_i.$$

we see that L_i has order dividing n in Λ_W/Λ_R.

Exercise 15.7. Show that L_i has order exactly n in Λ_W/Λ_R, so that $\Lambda_W/\Lambda_R \cong \mathbb{Z}/n\mathbb{Z}$.

From the above we can also say what the Weyl group is: the reflection in the hyperplane perpendicular to the root $L_i - L_j$ will exchange L_i and $L_j \in \mathfrak{h}^*$ and leave the other L_k alone, so that *the Weyl group* $\mathfrak{W}$ *is just the group* $\mathfrak{S}_n$, *acting as the symmetric group on the generators* L_i *of* $\mathfrak{h}^*$. Note that we have already verified that these automorphisms of $\mathfrak{h}^*$ do come from automorphisms of the whole Lie algebra $\mathfrak{sl}_n\mathbb{C}$ preserving $\mathfrak{h}$.

To continue, let us choose a direction, and describe the corresponding Weyl chamber. We can write our linear functional l as

$$l(\textstyle\sum a_i L_i) = \sum c_i a_i$$

with $\sum c_i = 0$; let us suppose that $c_1 > c_2 > \cdots > c_n$. The corresponding ordering of the roots will then be

$$R^+ = \{L_i - L_j\colon i < j\}$$

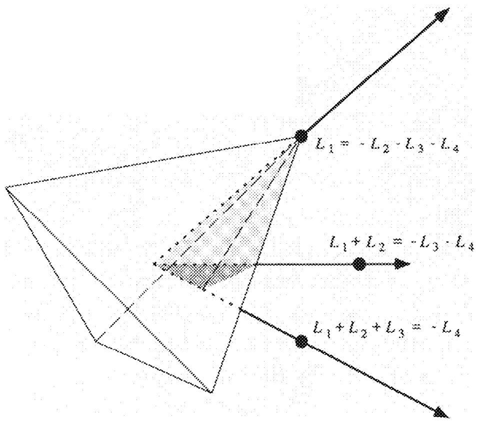
and

$$R^- = \{L_i - L_j: j < i\}.$$

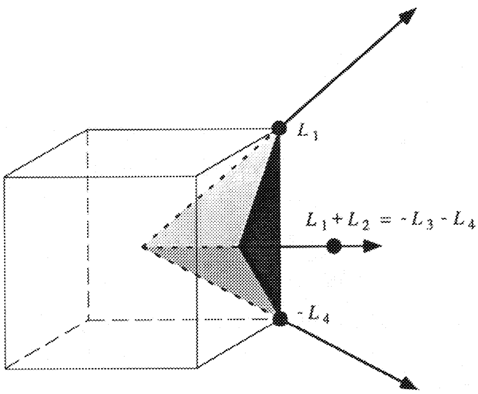
The primitive negative roots for this ordering are simply the roots $L_{i+1} - L_i$. (Note that the ordering of the roots depends only on the relative sizes of the c_i, so that the Weyl group acts simply transitively on the set of orderings.) The (closed) Weyl chamber associated to this ordering will then be the set

$$\mathscr{W} = \{\sum a_i L_i: a_1 \geq a_2 \geq \cdots \geq a_n\}.$$

One way to describe this geometrically is to say that if we take the barycentric subdivision of the faces of the simplex Δ, the Weyl chamber will be the cone over one $(n-2)$-simplex of the barycentric subdivision: e.g., in the case $n = 4$

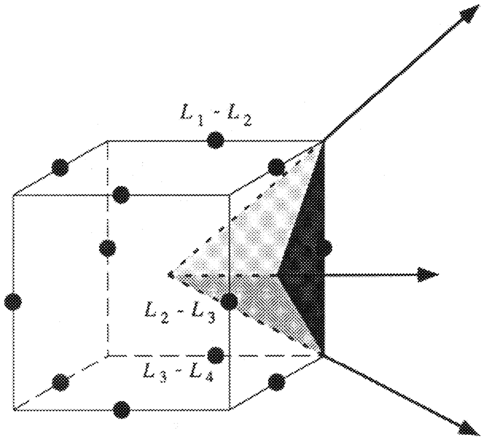


It may be easier to visualize the case $n = 4$ if we introduce the associated cubes: in terms of the cube with vertices at the points $\pm L_i$, we can draw the Weyl chamber as



(15.8)

Alternatively, in terms of the slightly larger cube with vertices at the points $\pm 2L_i$, we can draw $\mathscr{W}$ as



(15.9)

From the first of these pictures we see that the edges of the Weyl chamber are the rays generated by the vectors L_1, $L_1 + L_2$, and $L_1 + L_2 + L_3$; and that the faces of the Weyl chamber are the planes orthogonal to the primitive negative roots $L_2 - L_1$, $L_3 - L_2$, and $L_4 - L_3$. The picture in general is analogous: for $\mathfrak{sl}_n\mathbb{C}$, the Weyl chamber will be the cone over an $(n-2)$-simplex, with edges generated by the vectors

$$L_1, \quad L_1 + L_2, \quad L_1 + L_2 + L_3, \ldots, L_1 + \cdots + L_{n-1} = -L_n.$$

The faces of $\mathscr{W}$ will thus be the hyperplanes

$$\Omega_{L_i - L_{i+1}} = \{\sum a_j L_j \colon a_i = a_{i+1}\}$$

perpendicular to the primitive negative roots $L_{i+1} - L_i$.

Note the important phenomenon: the intersection of the closed Weyl chamber with the lattice Λ_W will be a free semigroup $\mathbb{N}^{n-1}$ generated by the fundamental weights $\omega_i = L_1 + \cdots + L_i$ occurring along the edges of the Weyl chamber. One aspect of its significance that is immediate is that it allows us to index the irreducible representations $\mathfrak{sl}_n\mathbb{C}$ nicely: for an arbitrary $(n-1)$-tuple of natural numbers $(a_1, \ldots, a_{n-1}) \in \mathbb{N}^{n-1}$ we will denote by $\Gamma_{a_1, \ldots, a_{n-1}}$ the irreducible representation of $\mathfrak{sl}_n\mathbb{C}$ with highest weight $a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1}) = (a_1 + \cdots + a_{n-1})L_1 + (a_2 + \cdots + a_{n-1})L_2 + \cdots + a_{n-1}L_{n-1}$:

$$\Gamma_{a_1, \ldots, a_{n-1}} = \Gamma_{a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1})}.$$

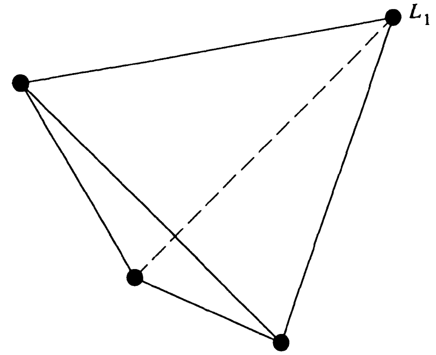
This also has the nice consequence that once we have located the irreducible representations $V^{(i)}$ with highest weight $L_1 + \cdots + L_i$, the general irreducible representation $\Gamma_{a_1,\ldots,a_{n-1}}$ with highest weight $\sum a_i(L_1 + \cdots + L_i)$ will occur inside the tensor product of symmetric powers

$$\mathrm{Sym}^{a_1}V^{(1)} \otimes \mathrm{Sym}^{a_2}V^{(2)} \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}V^{(n-1)}$$

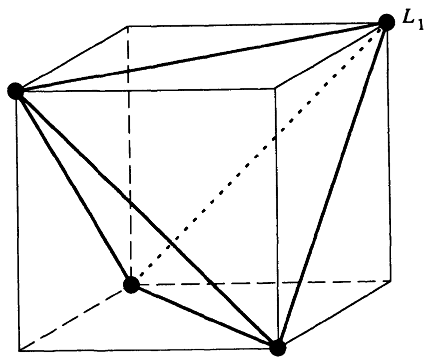
of these representations. Thus, the existence part of the basic Theorem 14.18 is reduced to finding the basic representations $V^{(i)}$; we will do this in due course, though at this point it is probably not too hard an exercise to guess what they are.

§15.2. Representations of $\mathfrak{sl}_4\mathbb{C}$ and $\mathfrak{sl}_n\mathbb{C}$

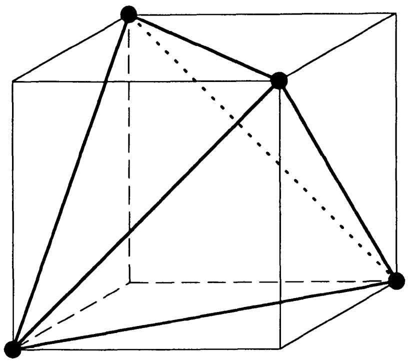
We begin as usual with the standard representation of $\mathfrak{sl}_4\mathbb{C}$ on $V = \mathbb{C}^4$. The standard basis vectors e_i of $\mathbb{C}^4$ are eigenvectors for the action of $\mathfrak{h}$, with eigenvalues L_i, so that the weight diagram looks like



or, with the reference cube drawn as well,

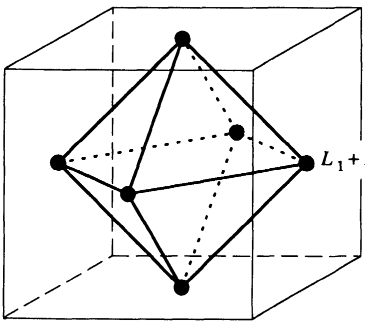


The dual representation V^* of course has weights $-L_i$ corresponding to the vectors of the dual basis e_i^* for V^*, so that the weight diagram, with its reference cube, looks like



Note that the highest weight for this representation is $-L_4$, which lies along the bottom edge of the Weyl chamber, as depicted in Diagram (15.8). Note also that the weights of the representation $\bigwedge^3 V$—the triple sums $L_1 + L_2 + L_3$, $L_1 + L_2 + L_4$, $L_1 + L_3 + L_4$, and $L_2 + L_3 + L_4$ of distinct weights of V—are the same as those of V^*, reflecting the isomorphism of these two representations.

This suggests that we look next at the second exterior power $\bigwedge^2 V$. This is a six-dimensional representation, with weights $L_i + L_j$ the pairwise sums of distinct weights of V; its weight diagram, in its reference cube, looks like



The diagram shows clearly that $\bigwedge^2 V$ is irreducible since it is not the nontrivial union of two configurations invariant under the Weyl group $\mathfrak{S}_4$ (and all weights occur with multiplicity 1). Note also that the weights are symmetric about the origin, reflecting the isomorphism of $\bigwedge^2 V$ with $(\bigwedge^2 V)^* = \bigwedge^2(V^*)$.

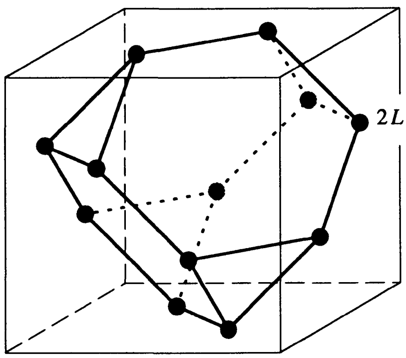
Note that the highest weight $L_1 + L_2$ of the representation $\bigwedge^2 V$ is the primitive vector along the front edge of the Weyl chamber $\mathscr{W}$ as pictured in Diagram (15.8). Now, we have already seen that the intersection of the closed

Weyl chamber with the weight lattice is a free semigroup generated by the primitive vectors along the three edges of $\mathscr{W}$—that is, every vector in $\mathscr{W} \cap \Lambda_W$ is a non-negative integral linear combination of the three vectors $L_1, L_1 + L_2$, and $L_1 + L_2 + L_3$. As we remarked at the end of the first section of this lecture, it follows that *we have proved the existence half of the general existence and uniqueness theorem* (14.18) *in the case of the Lie algebra* $\mathfrak{sl}_4\mathbb{C}$. Explicitly, since V, $\wedge^2 V$, and $\wedge^3 V = V^*$ have highest weight vectors with weights L_1, $L_1 + L_2$, and $L_1 + L_2 + L_3$, respectively, it follows that *the representation*

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b(\wedge^2 V) \otimes \mathrm{Sym}^c(\wedge^3 V)$$

contains a highest weight vector with weight $aL_1 + b(L_1 + L_2) + c(L_1 + L_2 + L_3)$, *and hence a copy of the irreducible representation* $\Gamma_{a,b,c}$ *with this highest weight.*

Let us continue our examination of representations of $\mathfrak{sl}_4\mathbb{C}$ with a pair of tensor products of the three basic representations: $V \otimes \wedge^2 V$ and $V \otimes \wedge^3 V$. As for the first of these, its weights are easy to find: they consist of the sums $2L_i + L_j$ (which occur once, as the sum of L_i and $L_i + L_j$) and $L_i + L_j + L_k$ (which occur three times). The diagram of these weights looks like



(We have drawn only the vertices of the convex hull of this diagram, thus omitting the weights $L_i + L_j + L_k$; they are located at the centers of the hexagonal faces of this polyhedron.)

Now, the representation $V \otimes \wedge^2 V$ cannot be irreducible, for at least a couple of reasons. First off, just by looking at weights, we see that the irreducible representation $W = \Gamma_{1,1,0}$ with highest weight $2L_1 + L_2$ can have multiplicity at most 2 on the weight $L_1 + L_2 + L_3$: by Observation 14.16, the weight space $W_{L_1+L_2+L_3}$ is generated by the images of the highest weight vector $v \in W_{2L_1+L_2}$ by successive applications of the primitive negative root spaces $\mathfrak{g}_{L_2-L_1}, \mathfrak{g}_{L_3-L_2}$, and $\mathfrak{g}_{L_4-L_3}$. But $L_1 + L_2 + L_3$ is uniquely expressible as a sum of $2L_1 + L_2$ and the primitive negative roots:

$$L_1 + L_2 + L_3 = 2L_1 + L_2 + (L_2 - L_1) + (L_3 - L_2);$$

so that $V_{L_1+L_2+L_3}$ is generated by the subspaces $\mathfrak{g}_{L_2-L_1}(\mathfrak{g}_{L_3-L_2}(v))$ and $\mathfrak{g}_{L_3-L_2}(\mathfrak{g}_{L_2-L_1}(v))$. We can in fact check that the representation $\Gamma_{1,1,0}$ takes on the weight $L_1 + L_2 + L_3$ with multiplicity 2 by writing out these generators explicitly and checking that they are independent: for example, we have

$$\begin{aligned}\mathfrak{g}_{L_2-L_1}(\mathfrak{g}_{L_3-L_2}(v)) &= \mathbb{C}\cdot E_{2,1}(E_{3,2}(e_1 \otimes (e_1 \wedge e_2))) \\ &= \mathbb{C}\cdot E_{2,1}(e_1 \otimes (e_1 \wedge e_3)) \\ &= \mathbb{C}\cdot(e_2 \otimes (e_1 \wedge e_3) + e_1 \otimes (e_2 \wedge e_3)).\end{aligned}$$

This is in fact what is called for in Exercise 15.10.

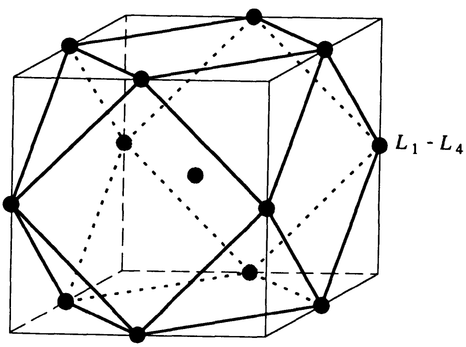
Alternatively, forgetting weights entirely, we can see from standard multilinear algebra that the representation $V \otimes \wedge^2 V$ cannot be irreducible: we have a natural map of representations

$$\varphi\colon V \otimes \wedge^2 V \to \wedge^3 V$$

which is obviously surjective. The kernel of this map is a representation with the same set of weights as $V \otimes \wedge^2 V$ (but taking on the weights $L_i + L_j + L_k$ with multiplicity 2 rather than 3), and so must contain the irreducible representation $\Gamma_{1,1,0}$ with highest weight $2L_1 + L_2$.

Exercise 15.10. Prove that the kernel of φ is indeed the irreducible representation $\Gamma_{1,1,0}$.

Finally, consider the tensor product $V \otimes \wedge^3 V$. This has weights $2L_i + L_k + L_l = L_i - L_j$, each occurring once, and 0, occurring four times. Its weight diagrams thus look like



This we may recognize as simply a direct sum of the adjoint representation with a copy of the trivial; this corresponds to the kernel and image of the obvious contraction (or trace) map

$$V \otimes \wedge^3 V = V \otimes V^* = \operatorname{Hom}(V, V) \to \mathbb{C}.$$

(Note that the adjoint representation is the irreducible representation with highest weight $2L_1 + L_2 + L_3$, or in other words the representation $\Gamma_{1,0,1}$.)

Exercise 15.11. Describe the weights of the representations $\operatorname{Sym}^n V$, and deduce that they are all irreducible.

Exercise 15.12. Describe the weights of the representations $\operatorname{Sym}^n(\wedge^2 V)$, and deduce that they are not irreducible. Describe maps

$$\varphi_n: \operatorname{Sym}^n(\wedge^2 V) \to \operatorname{Sym}^{n-2}(\wedge^2 V)$$

and show that the kernel of φ_n is the irreducible representation with highest weight $n(L_1 + L_2)$.

Exercise 15.13. The irreducible representation $\Gamma_{1,1,1}$ with highest weight $3L_1 + 2L_2 + L_3$ occurs as a subrepresentation of the tensor product $V \otimes \wedge^2 V \otimes \wedge^3 V$ lying in the kernel of each of the three maps

$$V \otimes \wedge^2 V \otimes \wedge^3 V \to \wedge^3 V \otimes \wedge^3 V$$
$$V \otimes \wedge^2 V \otimes \wedge^3 V \to \wedge^2 V \otimes \wedge^4 V \cong \wedge^2 V$$
$$V \otimes \wedge^2 V \otimes \wedge^3 V \cong V \otimes \wedge^2 V^* \otimes V^* \to V \otimes \wedge^3 V^* \cong V \otimes V$$

obtained by wedging two of the three factors. Is it equal to the intersection of these kernels? To test your graphic abilities, draw a diagram of the weights (ignoring multiplicities) of this representation.

Representations of $\mathfrak{sl}_n\mathbb{C}$

Once the case of $\mathfrak{sl}_4\mathbb{C}$ is digested, the case of the special linear group in general offers no surprises; the main difference in the general case is just the absence of pictures. Of course, the standard representation V of $\mathfrak{sl}_n\mathbb{C}$ has highest weight L_1, and similarly the exterior power $\wedge^k V$ is irreducible with highest weight $L_1 + \cdots + L_k$. It follows that the irreducible representation $\Gamma_{a_1,\ldots,a_{n-1}}$ with highest weight $(a_1 + \cdots + a_{n-1})L_1 + \cdots + a_{n-1}L_{n-1}$ will appear inside the tensor product

$$\operatorname{Sym}^{a_1} V \otimes \operatorname{Sym}^{a_2}(\wedge^2 V) \otimes \cdots \otimes \operatorname{Sym}^{a_{n-1}}(\wedge^{n-1} V),$$

demonstrating the existence theorem (14.18) for representations of $\mathfrak{sl}_n\mathbb{C}$.

Exercise 15.14. Verify that the exterior powers of the standard representations of $\mathfrak{sl}_n\mathbb{C}$ are indeed irreducible (though this is not necessary for the truth of the last sentence).

§15.3. Weyl's Construction and Tensor Products

At the end of the preceding section, we saw that the irreducible representation $\Gamma_{a_1,\ldots,a_{n-1}}$ of $\mathfrak{sl}_n\mathbb{C}$ with highest weight $(a_1 + \cdots + a_{n-1})L_1 + \cdots + a_{n-1}L_{n-1}$ will appear as a subspace of the tensor product

$$\mathrm{Sym}^{a_1}V \otimes \mathrm{Sym}^{a_2}(\wedge^2 V) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1}V),$$

or equivalently as a subspace of the dth tensor power $V^{\otimes d}$ of the standard representation V. The natural question is, how can we describe this subspace? We have seen the answer in one case already (two cases, if you count the trivial answer $\Gamma_a = \mathrm{Sym}^a V$ in the case $n = 2$): the representation $\Gamma_{a,b}$ of $\mathfrak{sl}_3\mathbb{C}$ can be realized as the kernel of the contraction map

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b(\wedge^2 V) \to \mathrm{Sym}^{a-1}V \otimes \mathrm{Sym}^{b-1}(\wedge^2 V).$$

This raises the question of whether the representation $\Gamma_{\mathbf{a}}$ can in general be described as a subspace of the tensor power $\bigotimes(\mathrm{Sym}^{a_i}(\wedge^i V))$ by intersecting kernels of such contraction/wedge product maps. Specifically, for i and j with $i + j \le n$ we can define maps

$$\begin{aligned}\mathrm{Sym}^{a_1}V &\otimes \mathrm{Sym}^{a_2}(\wedge^2 V) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1}V)\\ &\to \wedge^i V \otimes \wedge^j V \otimes \mathrm{Sym}^{a_1}V \otimes \cdots \otimes \mathrm{Sym}^{a_i-1}(\wedge^i V) \otimes \cdots\\ &\qquad \otimes \mathrm{Sym}^{a_j-1}(\wedge^j V) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\wedge^{n-1}V)\end{aligned}$$

and we have similar maps for $i < j$ with $i + j \ge n$ and i even with $2i \ge n$; there are likewise analogously defined maps in which we split off three or more factors. The representation $\Gamma_{a_1,\ldots,a_{n-1}}$ is in the kernel of all such maps; and we may ask whether the intersection of all such kernels is equal to $\Gamma_{\mathbf{a}}$.

The answer, it turns out, is no. (It is a worthwhile exercise to find an example of a representation $\Gamma_{\mathbf{a}}$ that cannot be realized in this way.) There is, however, another way of describing $\Gamma_{\mathbf{a}}$ as a subspace of $V^{\otimes d}$: in fact, we have already met these representations in Lecture 6, under the guise of *Schur functors* or *Weyl modules*. In fact, at the end of this lecture we will see how to describe them explicitly as subspaces of the above spaces $\bigotimes(\mathrm{Sym}^{a_i}(\wedge^i V))$. Recall that for $V = \mathbb{C}^n$ an n-dimensional vector space, and any partition

$$\lambda\colon \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0,$$

we can apply the Schur functor $\mathbb{S}_\lambda$ to V to obtain a representation $\mathbb{S}_\lambda V = \mathbb{S}_\lambda(\mathbb{C}^n)$ of $\mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$. If $d = \sum \lambda_i$, this was realized as

$$\mathbb{S}_\lambda V = V^{\otimes d}\cdot c_\lambda = V^{\otimes d} \otimes_{\mathbb{C}\mathfrak{S}_d} V_\lambda,$$

where c_λ is the Young symmetrizer corresponding to λ, and V_λ is the irreducible representation of $\mathfrak{S}_d$ corresponding to λ.

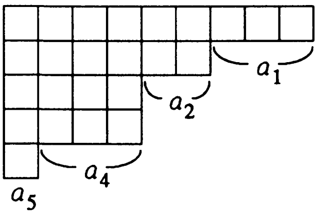
We saw in Lecture 6 that $\mathbb{S}_\lambda V$ is an irreducible representation of $\mathrm{GL}_n\mathbb{C}$. It follows immediately that $\mathbb{S}_\lambda V$ remains irreducible as a representation of $\mathrm{SL}_n\mathbb{C}$, since any element of $GL_n\mathbb{C}$ is a scalar multiple of an element of $SL_n\mathbb{C}$. In particular, it determines an irreducible representation of the Lie algebra $\mathfrak{sl}_n\mathbb{C}$.

Proposition 15.15. *The representation $\mathbb{S}_\lambda(\mathbb{C}^n)$ is the irreducible representation of $\mathfrak{sl}_n\mathbb{C}$ with highest weight $\lambda_1 L_1 + \lambda_2 L_2 + \cdots + \lambda_n L_n$.*

In particular, $\mathbb{S}_\lambda(\mathbb{C}^n)$ and $\mathbb{S}_\mu(\mathbb{C}^n)$ are isomorphic representations of $\mathfrak{sl}_n\mathbb{C}$ if and only if $\lambda_i - \mu_i$ is constant, independent of i. To relate this to our earlier notation, we may say that the irreducible representation $\Gamma_{a_1,\ldots,a_{n-1}}$ of $\mathfrak{sl}_n\mathbb{C}$ with highest weight $a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1})$ is obtained by applying the Schur functor $\mathbb{S}_\lambda$ to the standard representation V, where

$$\lambda = (a_1 + \cdots + a_{n-1}, a_2 + \cdots + a_{n-1}, \ldots, a_{n-1}, 0).$$

(If we want a unique Schur functor for each representation, we can restrict to those λ with $\lambda_n = 0$.) In terms of the Young diagram for λ, the coefficients $a_i = \lambda_i - \lambda_{i+1}$ are the differences of lengths of rows. For example, if $n = 6$,



is the Young diagram corresponding to $\Gamma_{3,2,0,3,1}$.

Proof of the proposition. In Theorem 6.3 we calculated that the trace of a diagonal matrix with entries $x_1, \ldots, x_n$ on $\mathbb{S}_\lambda(\mathbb{C}^n)$ is the Schur polynomial $S_\lambda(x_1, \ldots, x_n)$. By Equation (A.19), when the Schur polynomial is written out it takes the form

$$S_\lambda(x_1, \ldots, x_n) = M_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} M_\mu, \tag{15.16}$$

where M_μ is the sum of the monomial $X^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdot \ldots \cdot x_n^{\mu_n}$ and all distinct monomials obtained from it by permuting the variables, and the $K_{\lambda\mu}$ are certain non-negative integers called Kostka numbers. When $\mathbb{S}_\lambda(\mathbb{C}^n)$ is diagonalized with respect to the group of diagonal matrices in $GL_n(\mathbb{C})$, it is also diagonalized with respect to $\mathfrak{h} \subset \mathfrak{sl}_n(\mathbb{C})$. There is one monomial in the displayed equation for each one-dimensional eigenspace. The weights of $\mathbb{S}_\lambda(\mathbb{C}^n)$ as a representation of $\mathfrak{sl}_n(\mathbb{C})$ therefore consist of all

$$\mu_1 L_1 + \mu_2 L_2 + \cdots + \mu_n L_n,$$

each occurring as often as it does in the monomial X^μ in the polynomial

$S_\lambda(x_1, \ldots, x_n)$. Since the sum is over those partitions μ for which the first nonzero $\lambda_i - \mu_i$ is positive, the highest weight that appears is $\lambda_1 L_1 + \lambda_2 L_2 + \cdots + \lambda_n L_n$, which concludes the proof. [In fact one can describe an explicit basis of eigenvectors for $\mathbb{S}_\lambda(\mathbb{C}^n)$ which correspond to the monomials that appear in (15.16), cf. Problem 6.15 or Proposition 15.55.] □

In particular, we have (by Theorem 6.3) formulas for the dimension of the representation with given highest weight. Explicitly, one formula says that

$$\dim(\Gamma_{a_1, \ldots, a_{n-1}}) = \prod_{1 \le i < j \le n} \frac{(a_i + \cdots + a_{j-1}) + j - i}{j - i}. \tag{15.17}$$

As we saw in the proof, this proposition also gives the multiplicities of all weight spaces as the integers $K_{\lambda\mu}$ that appear in (15.16), which have a simple combinatorial description (p. 456): *the dimension of the weight space with weight* μ *in the representation* $\mathbb{S}_\lambda(\mathbb{C}^n)$ *is the number of ways one can fill the Young diagram of* λ *with* μ_1 1's, μ_2 2's, $\ldots$, μ_n n's, *in such a way that the entries in each row are nondecreasing and those in each column are strictly increasing.*

Exercise 15.18. Use the formula in case $n = 4$ to calculate the dimensions of the irreducible representations $\Gamma_{1,1,0}$ and $\Gamma_{1,1,1}$ of $\mathfrak{sl}_4\mathbb{C}$. In the former case, use this to redo Exercise 15.10; in the latter case, to do Exercise 15.13.

Exercise 15.19*. Use this formula to show that the dimension of the irreducible representation $\Gamma_{a,b}$ of $\mathfrak{sl}_3$ with highest weight $aL_1 + b(L_1 + L_2)$ is $(a + b + 1)(a + 1)(b + 1)/2$. This is the same as the dimension of the kernel of the contraction map

$$\iota_{a,b}\colon \mathrm{Sym}^a V \otimes \mathrm{Sym}^b V^* \to \mathrm{Sym}^{a-1} V \otimes \mathrm{Sym}^{b-1} V^*.$$

Use this to give another proof of the assertion made in Claim 13.4 that $\Gamma_{a,b}$ is this kernel.

Exercise 15.20*. As an application of the above formula, show that if V is the standard representation of $\mathfrak{sl}_n\mathbb{C}$, then the kernel of the wedge product map

$$V \otimes \Lambda^k V \to \Lambda^{k+1} V$$

is the irreducible representation $\Gamma_{1,0,\ldots,0,1,0,\ldots}$ with highest weight $2L_1 + L_2 + \cdots + L_k$; and that the irreducible representation $\Gamma_{k-1,1,0,\ldots}$ with highest weight $k \cdot L_1 + L_2$ is the kernel of the product map

$$V \otimes \mathrm{Sym}^k V \to \mathrm{Sym}^{k+1} V.$$

Exercise 15.21*. Show that the only nontrivial irreducible representations of $\mathfrak{sl}_n\mathbb{C}$ of dimension less than or equal to n are V and V^*.

One important consequence of the fact that the irreducible representations of $\mathfrak{sl}_n\mathbb{C}$ are obtained by applying Schur functors to the standard representation is that *identities among the Schur–Weyl functors give rise to identities among representations of* GL_n (and hence SL_n and $\mathfrak{sl}_n$), as we saw in Lecture 6. For example, the representation

$$\mathrm{Sym}^{\lambda_1}(V) \otimes \mathrm{Sym}^{\lambda_2}(V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_n}(V) \tag{15.22}$$

is a direct sum of representations $\mathbb{S}_\lambda(V) \oplus \bigoplus_\mu K_{\mu\lambda} \mathbb{S}_\mu(V)$, where $K_{\mu\lambda}$ is the coefficient described above. The particular application of this principle that we will use most frequently in the sequel, however, is the consequence that *one knows the decomposition of a tensor product of any two irreducible representations of* $\mathfrak{sl}_n\mathbb{C}$: specifically, the tensor power $\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V)$ decomposes into a direct sum of irreducible representations

$$\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V) = \bigoplus_\nu N_{\lambda\mu\nu} \mathbb{S}_\nu(V), \tag{15.23}$$

where the coefficients $N_{\lambda\mu\nu}$ are given by the *Littlewood–Richardson rule*, which is a formula in terms of the number of ways to fill the Young diagram between λ and ν with μ_1 1's, μ_2 2's, ..., μ_n n's, satisfying a certain combinatorial condition described in (A.8).

Exercise 15.24. Use the Littlewood–Richardson rule to show that the representation $\Gamma_{a_1+b_1,\ldots,a_{n-1}+b_{n-1}}$ occurs exactly once in the tensor product $\Gamma_{a_1,\ldots,a_{n-1}} \otimes \Gamma_{b_1,\ldots,b_{n-1}}$.

A special case of this is the analogue of Pieri's formula, which allows us to decompose the tensor product of an arbitrary irreducible representation with either $\mathrm{Sym}^k V = \Gamma_{k,0,\ldots,0}$ or the fundamental representation $\bigwedge^k V = \Gamma_{0,\ldots,1,0,\ldots,0}$, (where the 1 occurs in the kth place):

Proposition 15.25. (i) *The tensor product of* $\Gamma_{a_1,\ldots,a_{n-1}}$ *with* $\mathrm{Sym}^k V = \Gamma_{k,0,\ldots,0}$ *decomposes into a direct sum*:

$$\Gamma_{a_1,\ldots,a_{n-1}} \otimes \Gamma_{k,\ldots,0} = \bigoplus \Gamma_{b_1,\ldots,b_{n-1}},$$

the sum over all $(b_1, \ldots, b_{n-1})$ *for which there are non-negative integers* $c_1, \ldots, c_n$ *whose sum is* k, *with* $c_{i+1} \le a_i$ *for* $1 \le i \le n-1$, *and with* $b_i = a_i + c_i - c_{i+1}$ *for* $1 \le i \le n-1$.

(ii) *The tensor product of* $\Gamma_{a_1,\ldots,a_{n-1}}$ *with* $\bigwedge^k V = \Gamma_{0,\ldots,0,1,0,\ldots,0}$ *decomposes into a direct sum*:

$$\Gamma_{a_1,\ldots,a_{n-1}} \otimes \Gamma_{0,\ldots,0,1,0,\ldots,0} = \bigoplus \Gamma_{b_1,\ldots,b_{n-1}},$$

the sum over all $(b_1, \ldots, b_{n-1})$ *for which there is a subset* S *of* $\{1, \ldots, n\}$ *of cardinality* k, *such that if* $i \notin S$ *and* $i+1 \in S$, *then* $a_i > 0$, *with*

$$b_i = \begin{cases} a_i - 1 & \text{if } i \notin S \text{ and } i+1 \in S \\ a_i + 1 & \text{if } i \in S \text{ and } i+1 \notin S \\ a_i & \text{otherwise.} \end{cases}$$

Proof. This is simply a matter of translating the prescriptions of (6.8) and (6.9), which describe the decompositions in terms of adding boxes to the Young diagrams. In (i), the c_i are the number of boxes added to the ith row, and in (ii), S is the set of rows to which a box is added. □

Exercise 15.26. Verify the descriptions in Section 2 of this lecture of $V \otimes \wedge^2 V$ and $V \otimes \wedge^3 V$, where V is the standard representation of $\mathfrak{sl}_4\mathbb{C}$.

Exercise 15.27. Use Pieri's formula (with $n = 4$) twice to find the decomposition into irreducibles of $V \otimes \wedge^2 V \otimes \wedge^3 V$, where V is the standard representation of $\mathfrak{sl}_4\mathbb{C}$. Use this to redo Exercise 15.13.

Exercise 15.28. Use Pieri's formula to prove (13.5). You may also want to look around in Lecture 13 to see which other of the decompositions found there by hand may be deduced from these formulas.

Exercise 15.29. Verify that the statement of Exercise 15.20 follows directly from Pieri's formula.

In the following exercises, $V = \mathbb{C}^n$ is the standard representation of $\mathfrak{sl}_n\mathbb{C}$.

Exercise 15.30. Consider now tensor products of the form $\wedge^k V \otimes \wedge^l V$, with, say, $k \geq l$. Show that there is a natural map

$$\wedge^k V \otimes \wedge^l V \to \wedge^{k+1} V \otimes \wedge^{l-1} V$$

given by contraction with the element "trace" (or "identity") in $V \otimes V^* = \operatorname{End}(V)$. Explicitly, this map may be given by

$$(v_1 \wedge \cdots \wedge v_k) \otimes (w_1 \wedge \cdots \wedge w_l)$$
$$\mapsto \sum_{i=1}^{l} (-1)^i (v_1 \wedge \cdots \wedge v_k \wedge w_i) \otimes (w_1 \wedge \cdots \wedge \widehat{w_i} \wedge \cdots \wedge w_l).$$

What is the image of this map? Show that the kernel is the irreducible representation $\Gamma_{0,\ldots,0,1,0,\ldots,0,1,0,\ldots}$ with highest weight $2L_1 + \cdots + 2L_l + L_{l+1} + \cdots + L_k$.

Exercise 15.31*. Carry out an analysis similar to that of the preceding exercise for the maps

$$\operatorname{Sym}^k V \otimes \operatorname{Sym}^l V \to \operatorname{Sym}^{k+1} V \otimes \operatorname{Sym}^{l-1} V$$

defined analogously.

Exercise 15.32*. As a special case of Pieri's formula, we see that if V is the standard representation of $\mathfrak{sl}_n\mathbb{C}$, the tensor product

$$\begin{aligned}\Lambda^k V \otimes \Lambda^k V &= \bigoplus \mathbb{S}_{(2,\ldots,2,1,\ldots,1,0,\ldots)}(V)\\ &= \bigoplus \Gamma_{0,\ldots,0,1,0,\ldots,0,1,0,\ldots},\end{aligned}$$

where in the ith factor the 1's occur in the $(k-i)$th and $(k+i)$th places. At the same time, of course, we know that

$$\Lambda^k V \otimes \Lambda^k V = \mathrm{Sym}^2(\Lambda^k V) \oplus \Lambda^2(\Lambda^k V).$$

If we denote the ith term on the right-hand side of the first displayed equation for $\Lambda^k V \otimes \Lambda^k V$ by Θ_i, show that

$$\mathrm{Sym}^2(\Lambda^k V) = \bigoplus \Theta_{2i} \quad \text{and} \quad \Lambda^2(\Lambda^k V) = \bigoplus \Theta_{2i+1}.$$

Exercise 15.33*. As another special case of Pieri's formula, we see that the tensor product

$$\begin{aligned}\mathrm{Sym}^k V \otimes \mathrm{Sym}^k V &= \bigoplus \mathbb{S}_{(k+i,k-i)}(V)\\ &= \bigoplus \Gamma_{2i,k-i,0\ldots0}.\end{aligned}$$

At the same time, of course, we know that

$$\mathrm{Sym}^k V \otimes \mathrm{Sym}^k V = \mathrm{Sym}^2(\mathrm{Sym}^k V) \oplus \Lambda^2(\mathrm{Sym}^k V).$$

Which of the factors appearing in the first decomposition lie in $\mathrm{Sym}^2(\mathrm{Sym}^k V)$, and which in $\Lambda^2(\mathrm{Sym}^k V)$?

It follows from the Littlewood–Richardson rule that if λ, μ, and ν all have at most two rows, then the coefficient $N_{\lambda\mu\nu}$ is zero or one (and it is easy to say which occurs). In particular, for the Lie algebras $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$, the decomposition of the tensor product of two irreducible representations is always multiplicity free. Groups whose representations have this property, such as SU(2), SU(3), and SO(3) which are so important in physics, are called "simply reducible," cf. [Mack].

§15.4. Some More Geometry

Let V be an n-dimensional vector space, and $G(k, n) = G(k, V) = \mathrm{Grass}_k V$ the Grassmannian of k-planes in V. $\mathrm{Grass}_k V$ is embedded as a subvariety of the projective space $\mathbb{P}(\Lambda^k V)$ by the *Plücker embedding*:

$$\rho\colon \mathrm{Grass}_k V \hookrightarrow \mathbb{P}(\Lambda^k V)$$

sending the plane W spanned by vectors $v_1, \ldots, v_k$ to the alternating tensor $v_1 \wedge \cdots \wedge v_k$. Equivalently, noting that if $W \subset V$ is a k-dimensional subspace, then $\Lambda^k W$ is a line in $\Lambda^k V$, we may write this simply as

$$\rho\colon W \mapsto \Lambda^k W.$$

This embedding is compatible with the action of the general linear group:

$$\mathrm{PSL}_n\mathbb{C} = \mathrm{Aut}(\mathbb{P}(V)) = \{\sigma \in \mathrm{Aut}(\mathbb{P}(\wedge^k V)): \sigma(G(k, V)) = G(k, V)\}^{\circ}.$$

This follows from a fact in algebraic geometry ([Ha]): all automorphisms of the Grassmannian are induced by automorphisms of V, unless $n = 2k$, in which case we can choose an arbitrary isomorphism of V with V^* and compose these with the automorphism that takes W to $(\mathbb{C}^n/W)^*$. Here the superscript $\circ$ denotes the connected component of the identity. As in previous lectures, if we want symmetric powers to correspond to homogeneous polynomials on projective space, we should consider the dual situation: $G = \mathrm{Grass}^k V$ is the Grassmannian of k-dimensional *quotient* spaces of V, and the Plücker embedding embeds G in the projective space $\mathbb{P}(\wedge^k V^*)$ of one-dimensional quotients of $\wedge^k V$.

The space of all homogeneous polynomials of degree m on $\mathbb{P}(\wedge^k V^*)$ is naturally the symmetric power $\mathrm{Sym}^m(\wedge^k V)$. Let $I(G)_m$ denote the subspace of those polynomials of degree m on $\mathbb{P}(\wedge^k V^*)$ that vanish on G. Each $I(G)_m$ is a representation of $\mathfrak{sl}_n\mathbb{C}$:

$$0 \to I(G)_m \to \mathrm{Sym}^m(\wedge^k V) \to W_m \to 0,$$

where W_m denotes the restrictions to G of the polynomials of degree m on the ambient space $\mathbb{P}(\wedge^k V^*)$. We shall see later that W_m is the irreducible representation $\Gamma_{0,\dots,0,m,0,\dots}$ with highest weight $m(L_1 + \cdots + L_k)$ (the case $m = 2$ will be dealt with below). In the following discussion, we consider the problem of describing the quadratic part $I(G)_2$ of the ideal as a representation of $\mathfrak{sl}_n\mathbb{C}$.

Exercise 15.34. Consider the first case of a Grassmannian that is not a projective space, that is, $k = 2$. The ideal of the Grassmannian $G(2, V)$ of 2-planes in a vector space is easy to describe: a tensor $\varphi \in \wedge^2 V$ is decomposable if and only if $\varphi \wedge \varphi = 0$ (equivalently, if we think of φ as given by a skew-symmetric $n \times n$ matrix, if and only if the Pfaffians of symmetric 4×4 minors all vanish); and indeed the quadratic relations we get in this way generate the ideal of the Grassmannian. We, thus, have an isomorphism

$$I(G)_2 \cong \wedge^4 V$$

and correspondingly a decomposition into irreducibles

$$\mathrm{Sym}^2(\wedge^2 V) \cong \wedge^4 V \oplus \Gamma_{0,2,0,\dots,0},$$

where $\Gamma_{0,2,0,\dots,0}$ is, as above, the irreducible representation with highest weight $2(L_1 + L_2)$, cf. Exercise 15.32.

Exercise 15.35. When $k = 2$ and $n = 4$, G is a quadric hypersurface in $\mathbb{P}^5$, so polynomials vanishing on G are simply those divisible the quadratic polynomial that defines G. Deduce an isomorphism.

$$I(G)_m = \mathrm{Sym}^{m-2}(\wedge^2 V).$$

The first case of a Grassmannian that is not a projective space or of the form $G(2, V)$ is, of course, $G(3, 6)$, and this yields an interesting example.

Exercise 15.36. Let V be six dimensional. By examining weights, show that the space $I(G)_2$ of quadratic polynomials vanishing on the Grassmannian $G(3, V) \subset \mathbb{P}(\wedge^3 V)$ is isomorphic to the adjoint representation of $\mathfrak{sl}_6\mathbb{C}$, i.e., that we have a map

$$\varphi\colon \mathrm{Sym}^2(\wedge^3 V) \to V \otimes V^*$$

with image the space of traceless matrices.

Exercise 15.37. Find explicitly the map φ of the preceding exercise.

Exercise 15.38. Again, let V be six dimensional. Show that the representation $\mathrm{Sym}^4(\wedge^3 V)$ has a trivial direct summand, corresponding to the hypersurface in $\mathbb{P}(\wedge^3 V^*)$ dual to the Grassmannian $G = G(3, V) \subset \mathbb{P}(\wedge^3 V)$.

In general, the ideal $I(G) = \bigoplus I(G)_m$ is generated by the famous *Plücker equations*. These are homogeneous polynomials of degree two, and may be written down explicitly, cf. (15.53), [H-P], or [Ha]. In the following exercises, we will give a more intrinsic description of these relations, which will allow us to identify the space $I(G)_2$ they span as a representation on $\mathfrak{sl}_n\mathbb{C}$ (and to see the general pattern of which the above are special cases).

Exercise 15.39. For a given tensor $\Lambda \in \wedge^k V$, we introduce two associated subspaces:

$$W = \{v \in V\colon v \wedge \Lambda = 0\} \subset V$$

and

$$W^* = \{v^* \in V^*\colon v^* \wedge \Lambda^* = 0\} \subset V^*,$$

where, abusing notation slightly, Λ^* is the tensor Λ viewed as an element of $\wedge^k V = \wedge^{n-k} V^*$. Show that the dimensions of W and W^* are at most k and $n - k$, respectively, and that Λ is decomposable if and only if W has dimension k or W^* has dimension $n - k$; and deduce that Λ is decomposable if and only if the annihilator W' of W^* is equal to W.

Exercise 15.40. Now let $\Xi \in \wedge^{k+1} V^* = \wedge^{n-k-1} V$. Wedge product gives a map

$$\iota_\Xi\colon \wedge^k V \to \wedge^{n-1} V = V^*.$$

Using the preceding exercise, show that Λ is decomposable if and only if

$$\iota_\Xi(\Lambda) \wedge \Lambda = 0 \in \wedge^{k-1} V$$

for all $\Xi \in \wedge^{k+1} V^*$.

Exercise 15.41. Observe that in the preceding exercise we construct a map

$$\textstyle\bigwedge^{k+1}V^* \otimes \mathrm{Sym}^2(\bigwedge^k V) \to \bigwedge^{k-1}V,$$

or, by duality, a map

$$\textstyle\bigwedge^{k+1}V^* \otimes \bigwedge^{k-1}V^* \to \mathrm{Sym}^2(\bigwedge^k V^*) \qquad (15.42)$$

whose image is a vector space of quadrics on $\mathbb{P}(\bigwedge^k V)$ whose common zeros are exactly the locus of decomposable vectors, that is, the Grassmannian $G(k, V)$. Show that this image is exactly the span of the Plücker relations above.

Exercise 15.43. Show that the map (15.42) of the preceding exercise is just the dual of the map constructed in Exercise 15.30, with $k = l$ and restricted to the symmetric product. Combining this with the result of Exercise 15.32 (and assuming the statement that the Plücker relations do indeed span $I(G)_2$), deduce that in terms of the description

$$\textstyle\mathrm{Sym}^2(\bigwedge^k V) = \bigoplus \Theta_{2i}$$

of the symmetric square of $\bigwedge^k V$, we have

$$W_2 = \Theta_0 = \Gamma_{0,\dots,0,2,0,\dots}$$

(the irreducible representation with highest weight $2(L_1 + \cdots + L_k)$), and

$$I(G)_2 = \bigoplus_{i\geq 1} \Theta_{2i}.$$

Hard Exercise 15.44. Show that in the last equation the sub-direct sum

$$I(l) = \bigoplus_{i\geq l} \Theta_{2i}$$

is just the quadratic part of the ideal of the *restricted chordal variety* of the Grassmannian: that is, the union of the chords $\overline{LM}$ joining pairs of points in G corresponding to pairs of planes L and M meeting in a subspace of dimension at least $k - 2l + 1$. (Question: What is the actual zero locus of these quadrics?)

Exercise 15.45. Carry out an analysis similar to the above to relate the ideal of a Veronese variety $\mathbb{P}V^* \subset \mathbb{P}(\mathrm{Sym}^k V^*)$ to the decomposition given in Exercise 15.33 of $\mathrm{Sym}^2(\mathrm{Sym}^k V)$. For which k do the quadratic polynomials vanishing the Veronese give an irreducible representation?

Exercise 15.46. (For algebraic geometers and/or commutative algebraists.) Just as the group $\mathrm{PGL}_n\mathbb{C}$ acts on the ring S of polynomials on projective space $\mathbb{P}^N$, preserving the ideal of the Veronese variety, so it acts on that space of relations on the ideal (that is, inasmuch as the ideal is generated by quadrics, the kernel of the multiplication map $I_X(2) \otimes S \to S$), and likewise on the entire minimal resolution of the ideal of X. Show that this resolution has the form

$$\cdots \to R_2 \otimes S \to R_1 \otimes S \to I_X(2) \otimes S,$$

where all the R_i are finite-dimensional representations of $PGL_n\mathbb{C}$, and identify the representations R_i in the specific cases of

(i) the rational normal curve in $\mathbb{P}^3$,
(ii) the rational normal curve in $\mathbb{P}^4$, and
(iii) the Veronese surface in $\mathbb{P}^5$.

§15.5. Representations of $GL_n\mathbb{C}$

We have said that there is little difference between representations of $GL_n\mathbb{C}$ and those of the subgroup $SL_n\mathbb{C}$ of matrices of determinant 1. Our object here is to record the difference, which, naturally enough, comes from the determinant: if $V = \mathbb{C}^n$ is the standard representation, $\bigwedge^n V$ is trivial for $SL_n\mathbb{C}$ but not for $GL_n\mathbb{C}$. Similarly, V and $\bigwedge^{n-1} V^*$ are isomorphic for $SL_n\mathbb{C}$ but not for $GL_n\mathbb{C}$.

To relate representations of $SL_n\mathbb{C}$ and $GL_n\mathbb{C}$, we first need to define some representations of $GL_n\mathbb{C}$. To begin with, let D_k denote the one-dimensional representation of $GL_n\mathbb{C}$ given by the kth power of the determinant. When k is non-negative, $D_k = (\bigwedge^n V)^{\otimes k}$; D_{-k} is the dual $(D_k)^*$ of D_k. Next, note that the irreducible representations of $SL_n\mathbb{C}$ may be lifted to representations of $GL_n\mathbb{C}$ in two ways. First, for any index $\mathbf{a} = (a_1, \ldots, a_n)$ of length n we may take $\Phi_{\mathbf{a}}$ to be the subrepresentation of the tensor product

$$\mathrm{Sym}^{a_1} V \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}(\textstyle\bigwedge^{n-1} V) \otimes \mathrm{Sym}^{a_n}(\bigwedge^n V)$$

spanned by the highest weight vector with weight $a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_{n-1}(L_1 + \cdots + L_{n-1})$—that is, the vector

$$v = (e_1)^{a_1} \cdot (e_1 \wedge e_2)^{a_2} \cdot \ldots \cdot (e_1 \wedge \cdots \wedge e_n)^{a_n}.$$

This restricts to $SL_n\mathbb{C}$ to give the representation $\Gamma_{a'}$, where $a' = (a_1, \ldots, a_{n-1})$; taking different values of a_n amounts to tensoring the representation with different factors $\mathrm{Sym}^{a_n}(\bigwedge^n V) = (\bigwedge^n V)^{\otimes a_n} = D_{a_n}$. In particular, we have

$$\Phi_{a_1, \ldots, a_n + k} = \Phi_{a_1, \ldots, a_n} \otimes D_k,$$

which allows us to extend the definition of $\Phi_{\mathbf{a}}$ to indices a with $a_n < 0$: we simply set

$$\Phi_{a_1, \ldots, a_n} = \Phi_{a_1, \ldots, a_n + k} \otimes D_{-k}$$

for large k.

Alternatively, we may consider the Schur functor $\mathbb{S}_\lambda$ applied to the standard representation V of $GL_n\mathbb{C}$, where

$$\lambda = (a_1 + \cdots + a_n, a_2 + \cdots + a_n, \ldots, a_{n-1} + a_n, a_n).$$

We will denote this representation $\mathbb{S}_\lambda V$ of $GL_n\mathbb{C}$ by Ψ_λ; note that

$$\Psi_{\lambda_1 + k, \ldots, \lambda_n + k} = \Psi_{\lambda_1, \ldots, \lambda_n} \otimes D_k$$

which likewise allows us to define Ψ_λ for any index λ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, even if some of the λ_i are negative: we simply take

$$\Psi_{\lambda_1, \ldots, \lambda_n} = \Psi_{\lambda_1 + k, \ldots, \lambda_n + k} \otimes D_{-k}$$

for any sufficiently large k.

As is not hard to see, *the two representations* $\Phi_{\mathbf{a}}$ *and* Ψ_λ *are isomorphic as representations of* $\mathrm{GL}_n\mathbb{C}$: by §15.3 their restrictions to $\mathrm{SL}_n\mathbb{C}$ agree, so it suffices to check their restrictions to the center $\mathbb{C}^* \subset \mathrm{GL}_n\mathbb{C}$, where each acts by multiplication by $z^{\sum \lambda_i} = z^{\sum i a_i}$). It is even clearer that there are no coincidences among the $\Phi_{\mathbf{a}}$ (i.e., $\Phi_{\mathbf{a}}$ will be isomorphic to $\Phi_{\mathbf{a}'}$ if and only if $\mathbf{a} = \mathbf{a}'$): if $\Phi_{\mathbf{a}} \cong \Phi_{\mathbf{a}'}$, we must have $a_i = a_i'$ for $i = 1, \ldots, n-1$, so the statement follows from the nontriviality of D_k for $k \neq 0$. Thus, to complete our description of the irreducible finite-dimensional representations of $\mathrm{GL}_n\mathbb{C}$, we just have to check that we have found them all. We may then express the completed result as

Proposition 15.47. *Every irreducible complex representation of* $\mathrm{GL}_n\mathbb{C}$ *is isomorphic to* Ψ_λ *for a unique index* $\lambda = \lambda_1, \ldots, \lambda_n$ *with* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ *(equivalently, to* $\Phi_{\mathbf{a}}$ *for a unique index* $\mathbf{a} = a_1, \ldots, a_n$ *with* $a_1, \ldots, a_{n-1} \geq 0$*).*

Proof. We start by going back to the corresponding Lie algebras. The scalar matrices form a one-dimensional ideal $\mathbb{C}$ in $\mathfrak{gl}_n\mathbb{C}$, and in fact $\mathfrak{gl}_n\mathbb{C}$ is a product of Lie algebras:

$$\mathfrak{gl}_n\mathbb{C} = \mathfrak{sl}_n\mathbb{C} \times \mathbb{C}. \tag{15.48}$$

In particular, $\mathbb{C}$ is the radical of $\mathfrak{gl}_n\mathbb{C}$, and $\mathfrak{sl}_n\mathbb{C}$ is the semisimple part. It follows from Proposition 9.17 that every irreducible representation of $\mathfrak{gl}_n\mathbb{C}$ is a tensor product of an irreducible representation of $\mathfrak{sl}_n\mathbb{C}$ and a one-dimensional representation. More precisely, let $W_\lambda = \mathbb{S}_\lambda(\mathbb{C}^n)$ be the representation of $\mathfrak{sl}_n\mathbb{C}$ determined by the partition λ (extended to $\mathfrak{sl}_n\mathbb{C} \times \mathbb{C}$ by making the second factor act trivially). For $w \in \mathbb{C}$, let $L(w)$ be the one-dimensional representation of $\mathfrak{sl}_n\mathbb{C} \times \mathbb{C}$ which is zero on the first factor and multiplication by w on the second; the proof of Proposition 9.17 shows that any irreducible representation of $\mathfrak{sl}_n\mathbb{C} \times \mathbb{C}$ is isomorphic to a tensor product $W_\lambda \otimes L(w)$. The same is therefore true for the simply connected[1] group $\mathrm{SL}_n\mathbb{C} \times \mathbb{C}$ with this Lie algebra.

We write $\mathrm{GL}_n\mathbb{C}$ as a quotient modulo a discrete subgroup of the center of $\mathrm{SL}_n\mathbb{C} \times \mathbb{C}$:

$$1 \to \mathrm{Ker}(\rho) \to \mathrm{SL}_n\mathbb{C} \times \mathbb{C} \xrightarrow{\rho} \mathrm{GL}_n\mathbb{C} \to 1, \tag{15.49}$$

where $\rho(g \times z) = e^z \cdot g$, so the kernel of ρ is generated by $e^s \cdot I \times (-s)$, where $s = 2\pi i/n$.

Our task is simply to see which of the representations $W_\lambda \otimes L(w)$ of $\mathrm{SL}_n\mathbb{C} \times \mathbb{C}$ are trivial on the kernel of ρ. Now $e^s \cdot I$ acts on $\mathbb{S}_\lambda\mathbb{C}^n$ by multi-

[1] For a proof that $\mathrm{SL}_n\mathbb{C}$ is simply connected, see §23.1.

plication by e^{sd}, where $d = \sum \lambda_i$; indeed, this is true on the entire representation $(\mathbb{C}^n)^{\otimes d}$ which contains $\mathbb{S}_\lambda\mathbb{C}^n$. And $-s$ acts on $L(w)$ by multiplication by e^{-sw}, so $e^s \cdot I \times (-s)$ acts on the tensor product by multiplication by e^{sd-sw}. The tensor product is, therefore, trivial on the kernel of ρ precisely when $sd - sw \in 2\pi i\mathbb{Z}$, i.e., when

$$w = \sum \lambda_i + kn$$

for some integer k.

We claim finally that any representation $W_\lambda \otimes L(w)$ satisfying this condition is the pullback via ρ of a representation Ψ on $\mathrm{GL}_n\mathbb{C}$. In fact, it is not hard to see that it is the pullback of the representation $\Psi_{\lambda_1+k,\dots,\lambda_n+k}$: the two clearly restrict to the same representation on $\mathrm{SL}_n\mathbb{C}$, and their restrictions to $\mathbb{C}$ are just multiplication by $e^{wz} = e^{(\sum \lambda_i + nk)z}$. □

Exercise 15.50. Show that the dual of the representation Ψ_λ which is isomorphic to $\mathbb{S}_\lambda(V^*)$ is the representation $\Psi_{(-\lambda_n,\dots,-\lambda_1)}$.

Exercise 15.51*. Show that if $\rho: \mathrm{GL}_n\mathbb{C} \to \mathrm{GL}(W)$ is a representation (assumed to be holomorphic), then W decomposes into a direct sum of irreducible representations.

Exercise 15.52*. Show that the Hermite reciprocity isomorphism of Exercise 11.34 is an isomorphism over $\mathrm{GL}_2\mathbb{C}$, not just over $\mathrm{SL}_2\mathbb{C}$.

More Remarks on Weyl's Construction

We close out this lecture by looking once more at the Weyl construction of these representations of $\mathrm{GL}(V)$. This will include a realization "by generators and relations," as well as giving a natural basis for each representation. First, it may be illuminating—and it will be useful later—to look more closely at how $\mathbb{S}_\lambda V$ sits in $V^{\otimes d}$. We want to realize $\mathbb{S}_\lambda V$ as a subspace of the subspace

$$\mathrm{Sym}^{a_k}(\Lambda^k V) \otimes \mathrm{Sym}^{a_{k-1}}(\Lambda^{k-1} V) \otimes \cdots \otimes \mathrm{Sym}^{a_1}(V) \subset V^{\otimes d},$$

where a_i is the number of columns of the Young diagram of λ of length i (and k is the number of rows). This space is embedded in $V^{\otimes d}$ in the natural way: from left to right, a factor $\mathrm{Sym}^a(\Lambda^b V)$ is embedded in the corresponding $V^{\otimes ab}$ by mapping a symmetric product of exterior products

$$(v_{1,1} \wedge v_{2,1} \wedge \cdots \wedge v_{b,1}) \cdot (v_{1,2} \wedge v_{2,2} \wedge \cdots \wedge v_{b,2}) \cdot \ldots \cdot (v_{1,a} \wedge v_{2,a} \wedge \cdots \wedge v_{b,a})$$

to

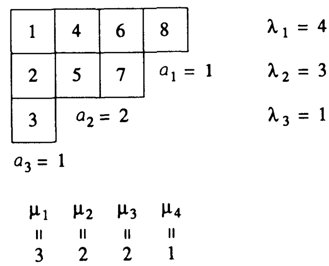
$$\sum \mathrm{sgn}(q)(v_{q_1(1),p(1)} \otimes \cdots \otimes v_{q_1(b),p(1)}) \otimes \cdots \otimes (v_{q_a(1),p(a)} \otimes \cdots \otimes v_{q_a(b),p(a)}),$$

the sum over $p \in \mathfrak{S}_a$ and $q = (q_1, \dots, q_a) \in \mathfrak{S}_b \times \cdots \times \mathfrak{S}_b$. In other words, one first symmetrizes by permuting columns of the same length, and then performs an alternating symmetrizer on each column.

Letting $\mathbf{a} = (a_1, \ldots, a_k)$, let $A^{\mathbf{a}}(V)$ denote this tensor product of symmetric powers of exterior powers, i.e., set

$$A^{\mathbf{a}}V = \mathrm{Sym}^{a_k}(\textstyle\bigwedge^k V) \otimes \mathrm{Sym}^{a_{k-1}}(\textstyle\bigwedge^{k-1} V) \otimes \cdots \otimes \mathrm{Sym}^{a_1}(V).$$

We want to realize $\mathbb{S}_\lambda V$ as a subspace of $A^{\mathbf{a}}V$. To do this we use the construction of $\mathbb{S}_\lambda V$ as $V^{\otimes d} \cdot c_\lambda$, where c_λ is a Young symmetrizer; to get compatibility with the embedding of $A^{\mathbf{a}}V$ we have just made, we use the tableau which numbers the columns from top to bottom, then left to right.



We take $\mu = \lambda' = (\mu_1 \geq \cdots \geq \mu_l > 0)$ to be the conjugate of λ. The symmetrizer c_λ is a product $a_\lambda \cdot b_\lambda$, where $a_\lambda = \sum e_p$, the sum over all p in the subgroup $P = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k}$ of $\mathfrak{S}_d$ preserving the rows, $b_\lambda = \sum \mathrm{sgn}(q)q$, the sum over the subgroup $Q = \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_l}$ preserving the columns, as described in Lecture 4. The symmetrizing by rows can be done in two steps as follows. There is a subgroup

$$R = \mathfrak{S}_{a_k} \times \cdots \times \mathfrak{S}_{a_1}$$

of P, which consists of permutations that move all entries of each column to the same position in some column of the same length; in other words, permutations in R are determined by permuting columns which have the same length. (In the illustration, $R = \{1, (46)(57)\}$.) Set

$$a'_\lambda = \sum_{r \in R} e_r \quad \text{in } \mathbb{C}\mathfrak{S}_d.$$

Now if we define a''_λ to be $\sum e_p$, where the sum is over any set of representatives in P for the left cosets P/R, then the row symmetrizer a_λ is the product of a''_λ and a'_λ. So

$$\mathbb{S}_\lambda(V) = (V^{\otimes d} \cdot a''_\lambda) \cdot a'_\lambda \cdot b_\lambda.$$

The point is that, by what we have just seen,

$$V^{\otimes d} \cdot a'_\lambda \cdot b_\lambda = A^{\mathbf{a}}V.$$

Since $V^{\otimes d}\cdot a_\lambda''$ is a subspace of $V^{\otimes d}$, its image $\mathbb{S}_\lambda(V)$ by $a_\lambda'\cdot b_\lambda$ is a subspace of $A^{\mathbf{a}}(V)$, as we claimed.

There is a simple way to construct all the representations $\mathbb{S}_\lambda V$ of $\mathrm{GL}(V)$ at once. In fact, the direct sum of all the representations $\mathbb{S}_\lambda V$, over all (non-negative) partitions λ, can be made into a commutative, graded ring, which we denote by $\mathbb{S}^\bullet$ or $\mathbb{S}^\bullet(V)$, with simple generators and relations. This is similar to the fact that the symmetric algebra $\mathrm{Sym}^\bullet V = \bigoplus \mathrm{Sym}^k V$ and the exterior algebra $\bigwedge^\bullet V = \bigoplus \bigwedge^k V$ are easier to describe than the individual graded pieces, and it has some of the similar advantages for studying all the representations at once. This algebra has appeared and reappeared frequently, cf. [H-P]; the construction we give is essentially that of Towber [Tow1].

To construct $\mathbb{S}^\bullet(V)$, start with the symmetric algebra on the sum of all the positive exterior products of V: set

$$\begin{aligned} A^\bullet(V) &= \mathrm{Sym}^\bullet(V \oplus \textstyle\bigwedge^2 V \oplus \bigwedge^3 V \oplus \cdots \oplus \bigwedge^n V) \\ &= \bigoplus_{a_1,\ldots,a_n} \mathrm{Sym}^{a_n}(\textstyle\bigwedge^n V) \otimes \cdots \otimes \mathrm{Sym}^{a_2}(\bigwedge^2 V) \otimes \mathrm{Sym}^{a_1}(V), \end{aligned}$$

the sum over all n-tuples $a_1, \ldots, a_n$ of non-negative integers. So $A^\bullet(V)$ is the direct sum of the $A^{\mathbf{a}}(V)$ just considered. The ring $\mathbb{S}^\bullet = \mathbb{S}^\bullet(V)$ is defined to be the quotient of this ring $A^\bullet(V)$ modulo the graded, two-sided ideal $I^\bullet$ generated by all elements ("Plücker relations") of the form

$$\begin{aligned} &(v_1 \wedge \cdots \wedge v_p)\cdot(w_1 \wedge \cdots \wedge w_q) \\ &\quad - \sum_{i=1}^{p} (v_1 \wedge \cdots \wedge v_{i-1} \wedge w_1 \wedge v_{i+1} \wedge \cdots \wedge v_p)\cdot(v_i \wedge w_2 \wedge \cdots \wedge w_q) \end{aligned} \tag{15.53}$$

for all $p \geq q \geq 1$ and all $v_1, \ldots, v_p, w_1, \ldots, w_q \in V$. (If $p = q$, this is an element of $\mathrm{Sym}^2(\bigwedge^p V)$; if $p > q$, it is in $\bigwedge^p V \otimes \bigwedge^q V = \mathrm{Sym}^1(\bigwedge^p V) \otimes \mathrm{Sym}^1(\bigwedge^q V)$. Note that the multiplication in $\mathbb{S}^\bullet(V)$ comes entirely from its being a symmetric algebra and does not involve the wedge products in $\bigwedge^\bullet V$.)

Exercise 15.54*. Show that $I^\bullet$ contains all elements of the form

$$\begin{aligned} &(v_1 \wedge \cdots \wedge v_p)\cdot(w_1 \wedge \cdots \wedge w_q) \\ &\quad - \sum (v_1 \wedge \cdots \wedge w_1 \wedge \cdots \wedge w_r \wedge \cdots \wedge v_p) \\ &\qquad \cdot(v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_r} \wedge w_{r+1} \wedge \cdots \wedge w_q) \end{aligned}$$

for all $p \geq q \geq r \geq 1$ and all $v_1, \ldots, v_p, w_1, \ldots, w_q \in V$, where the sum is over all $1 \leq i_1 < i_2 < \cdots < i_r \leq p$, and the elements $w_1, \ldots, w_r$ are inserted at the corresponding places in $v_1 \wedge \cdots \wedge v_p$.

Remark. You can avoid this exercise by simply taking the elements in the exercise as defining generators for the ideal $I^\bullet$. When $p = q = r$, the calculation of Exercise 15.54 shows that the relation $(v_1 \wedge \cdots \wedge v_p)\cdot(w_1 \wedge \cdots \wedge w_p) = (w_1 \wedge \cdots \wedge w_p)\cdot(v_1 \wedge \cdots \wedge v_p)$ follows from the generating equations for $I^{\bullet}$. In particular, this commutativity shows that one could define $\mathbb{S}^{\bullet}(V)$ to be the full tensor algebra on $V \oplus \bigwedge^2 V \oplus \cdots \oplus \bigwedge^n V$ modulo the ideal generated by the same generators.

The algebra $\mathbb{S}^{\bullet}(V)$ is the direct sum of the images $\mathbb{S}^{\mathbf{a}}(V)$ of the summands $A^{\mathbf{a}}(V)$. Let $e_1, \ldots, e_n$ be a basis for V. We will construct a basis for $\mathbb{S}^{\mathbf{a}}(V)$, with a basis element e_T for every semistandard tableau T on the partition λ which corresponds to $\mathbf{a}$. Recall that a semistandard tableau is a numbering of the boxes of the Young diagram with the integers $1, \ldots, n$, in such a way that the entries in each row are nondecreasing, and the entries in each column are strictly increasing. Let $T(i, j)$ be the entry of T in the ith row and the jth column. Define e_T to be the image in $\mathbb{S}^{\mathbf{a}}(V)$ of the element

$$\prod_{j=1}^{l} e_{T(1,j)} \wedge e_{T(2,j)} \wedge \cdots \wedge e_{T(\mu_j,j)} \in \mathrm{Sym}^{a_n}(\textstyle\bigwedge^n V) \otimes \cdots \otimes \mathrm{Sym}^{a_1}(V),$$

i.e., wedge together the basis elements corresponding to the entries in the columns, and multiply the results in $\mathbb{S}^{\bullet}(V)$.

Proposition 15.55. (1) *The projection from $A^{\mathbf{a}}(V)$ to $\mathbb{S}^{\mathbf{a}}(V)$ maps the subspace $\mathbb{S}_\lambda(V)$ isomorphically onto $\mathbb{S}^{\mathbf{a}}(V)$.*

(2) *The e_T for T a semistandard tableau on λ form a basis for $\mathbb{S}^{\mathbf{a}}(V)$.*

Proof. We show first that the elements e_T span $\mathbb{S}^{\mathbf{a}}(V)$. It is clear that the e_T span if we allow all tableaux T that number the boxes of λ with integers between 1 and n with strictly increasing columns, for such elements span before dividing by the ideal $I^{\bullet}$. We order such tableaux by listing their entries column by column, from left to right and top to bottom, and using the reverse lexicographic order: $T' > T$ if the last entry where they differ has a larger entry for T' than for T. If T is not semistandard, there will be two successive columns of T, say the jth and $(j+1)$st, in which we have $T(r, j) > T(r, j+1)$ for some r. It suffices to show how to use relations in $I^{\bullet}$ to write e_T as a linear combination of elements $e_{T'}$ with $T' > T$. For this we use the relation in Exercise 15.54, with $v_i = e_{T(i,j)}$ for $1 \le i \le p = \mu_j$, and $w_i = e_{T(i,j+1)}$ for $1 \le i \le q = \mu_{j+1}$, to interchange the first r of the $\{w_i\}$ with subsets of r of the $\{v_i\}$. The terms on the right-hand side of the relation will all correspond to tableaux T' in which the r first entries in the $(j+1)$st column of T are replaced by r of the enties in the jth column, and are not otherwise changed beyond the jth column. All of these are larger than T in the ordering, which proves the assertion.

It is possible to give a direct proof that the e_T corresponding to semistandard tableaux T are linearly independent (see [Tow1]), but we can get by with less. Among the semistandard tableaux on λ there is a smallest one T_0 whose ith row is filled with the integer i. We need to know that e_{T_0} is not zero in $\mathbb{S}^{\bullet}$. This is easy to see directly. In fact, the relations among the e_T in $I^{\bullet} \cap A^{\mathbf{a}}(V)$ are spanned by those obtained by substituting r elements from some column of some T to an earlier column, as in the preceding paragraph. Such will never involve the generator e_{T_0} unless the T that is used is T_0, and in this case, the resulting element of $I^{\bullet}$ is zero. Since e_{T_0} occurs in no nontrivial relation, its image in $\mathbb{S}^{\bullet}$ cannot vanish.

Since e_{T_0} comes from $\mathbb{S}_\lambda(V)$, it follows that the projection from $\mathbb{S}_\lambda(V)$ to $\mathbb{S}^{\mathbf{a}}(V)$ is not zero. Since this projection is a mapping of representations of $\mathrm{SL}(V)$, it follows that $\mathbb{S}^{\mathbf{a}}(V)$ must contain a copy of the irreducible representation $\mathbb{S}_\lambda(V)$. We know from Theorem 6.3 and Exercise A.31 that the dimension of $\mathbb{S}_\lambda(V)$ is the number of semistandard tableaux on λ. Since we have proved that the dimension of $\mathbb{S}^{\mathbf{a}}(V)$ is at most this number, the projection from $\mathbb{S}_\lambda(V)$ to $\mathbb{S}^{\mathbf{a}}(V)$ must be surjective, and since $\mathbb{S}_\lambda(V)$ is irreducible, it must be injective as well, and the e_T for T a semistandard tableau on λ must form a basis, as asserted. $\square$

Note that this proposition gives another description of the representations $\mathbb{S}_\lambda(V)$, as the quotient of the space $A^{\mathbf{a}}(V)$ by the subspace generated by the "Plücker" relations (15.53).

Exercise 15.56. Show that, if the factor $\bigwedge^n V$ is omitted from the construction, the resulting algebra is the direct sum of all irreducible representations of $\mathrm{SL}(V) = \mathrm{SL}_n\mathbb{C}$.

It is remarkable that all the representations $\mathbb{S}_\lambda(\mathbb{C}^n)$ of $\mathrm{GL}_n\mathbb{C}$ were written down by Deruyts (following Clebsch) a century ago, before representation theory was born, as in the following exercise.

Exercise 15.57*. Let $X = (x_{i,j})$ be an $n \times n$ matrix of indeterminates. The group $G = \mathrm{GL}_n\mathbb{C}$ acts on the polynomial ring $\mathbb{C}[x_{i,j}]$ by $g \cdot x_{i,j} = \sum_{k=1}^n a_{k,i} x_{k,j}$ for $g = (a_{i,j}) \in \mathrm{GL}_n\mathbb{C}$. For any tableau T on the Young diagram of λ consisting of the integers from 1 to n, strictly increasing in the columns, let e_T be the product of minors constructed from X, one for each column, as follows: if the column of T has length μ_j, form the minor using the first μ_j columns, and use the rows that are numbered by the entries of the column of T. Let D_λ be the subspace of $\mathbb{C}[x_{i,j}]$ spanned by these e_T, where d is the number partitioned by λ. Show that: (i) D_λ is preserved by $\mathrm{GL}_n\mathbb{C}$; (ii) the e_T, where T is semistandard, form a basis for D_λ; (iii) D_λ is isomorphic to $\mathbb{S}_\lambda(\mathbb{C}^n)$.

LECTURE 16

Symplectic Lie Algebras

In this lecture we do for the symplectic Lie algebras exactly what we did for the special linear ones in §15.1 and most of §15.2: we will first describe in general the structure of a symplectic Lie algebra (that is, give a Cartan subalgebra, find the roots, describe the Killing form, and so on). We will then work out in some detail the representations of the specific algebra $\mathfrak{sp}_4\mathbb{C}$. As in the case of the corresponding analysis of the special linear Lie algebras, this is completely elementary.

§16.1. The Structure of $\mathrm{Sp}_{2n}\mathbb{C}$ and $\mathfrak{sp}_{2n}\mathbb{C}$

Let V be a $2n$-dimensional complex vector space, and

$$Q: V \times V \to \mathbb{C},$$

a nondegenerate, skew-symmetric bilinear form on V. The symplectic Lie group $\mathrm{Sp}_{2n}\mathbb{C}$ is then defined to be the group of automorphisms A of V preserving Q—that is, such that $Q(Av, Aw) = Q(v, w)$ for all $v, w \in V$—and the symplectic Lie algebra $\mathfrak{sp}_{2n}\mathbb{C}$ correspondingly consists of endomorphisms $A: V \to V$ satisfying

$$Q(Av, w) + Q(v, Aw) = 0$$

for all v and $w \in V$. Clearly, the isomorphism classes of the abstract group and Lie algebra do not depend on the particular choice of Q; but in order to be able to write down elements of both explicitly we will, for the remainder of our discussion, take Q to be the bilinear form given, in terms of a basis $e_1, \ldots,$

e_{2n} for V, by

$$Q(e_i, e_{i+n}) = 1,$$
$$Q(e_{i+n}, e_i) = -1,$$

and

$$Q(e_i, e_j) = 0 \quad \text{if } j \neq i \pm n.$$

The bilinear form Q may be expressed as

$$Q(x, y) = {}^t x \cdot M \cdot y,$$

where M is the $2n \times 2n$ matrix given in block form as

$$M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix};$$

the group $\mathrm{Sp}_{2n}\mathbb{C}$ is thus the group of $2n \times 2n$ matrices A satisfying

$$M = {}^t A \cdot M \cdot A$$

and the Lie algebra $\mathfrak{sp}_{2n}\mathbb{C}$ correspondingly the space of matrices X satisfying the relation

$$ {}^t X \cdot M + M \cdot X = 0. \tag{16.1}$$

Writing a $2n \times 2n$ matrix X in block form as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

we have

$$ {}^t X \cdot M = \begin{pmatrix} -{}^t C & {}^t A \\ -{}^t D & {}^t B \end{pmatrix}$$

and

$$M \cdot X = \begin{pmatrix} C & D \\ -A & -B \end{pmatrix}$$

so that this relation is equivalent to saying that *the off-diagonal blocks B and C of X are symmetric, and the diagonal blocks A and D of X are negative transposes of each other.*

With this said, there is certainly an obvious candidate for Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{sp}_{2n}\mathbb{C}$, namely the subalgebra of matrices diagonal in this representation; in fact, this works, as we shall see shortly. The subalgebra $\mathfrak{h}$ is thus spanned by the n $2n \times 2n$ matrices $H_i = E_{i,i} - E_{n+i,n+i}$ whose action on V is to fix e_i, send e_{n+i} to its negative, and kill all the remaining basis vectors; we will correspondingly take as basis for the dual vector space $\mathfrak{h}^*$ the dual basis L_j, where $\langle L_j, H_i \rangle = \delta_{i,j}$.

We have already seen how the diagonal matrices act on the algebra of all matrices, so that it is easy to describe the action of $\mathfrak{h}$ on $\mathfrak{g}$. For example, for

$1 \le i, j \le n$ the matrix $E_{i,j} \in \mathfrak{gl}_{2n}\mathbb{C}$ is carried into itself under the adjoint action of H_i, into minus itself by the action of H_j, and to 0 by all the other H_k; and the same is true of the matrix $E_{n+j,n+i}$. The element

$$X_{i,j} = E_{i,j} - E_{n+j,n+i} \in \mathfrak{sp}_{2n}\mathbb{C}$$

is thus an eigenvector for the action of $\mathfrak{h}$, with eigenvalue $L_i - L_j$. Similarly, for $i \neq j$ we see that the matrices $E_{i,n+j}$ and $E_{j,n+i}$ are carried into themselves by H_i and H_j and killed by all the other H_k; and likewise $E_{n+i,j}$ and $E_{n+j,i}$ are each carried into their negatives by H_i and H_j and killed by the others. Thus, the elements

$$Y_{i,j} = E_{i,n+j} + E_{j,n+i}$$

and

$$Z_{i,j} = E_{n+i,j} + E_{n+j,i}$$

are eigenvectors for the action of $\mathfrak{h}$, with eigenvalues $L_i + L_j$ and $-L_i - L_j$, respectively. Finally, when $i = j$ the same calculation shows that $E_{i,n+i}$ is doubled by H_i and killed by all other H_j; and likewise $E_{n+i,i}$ is sent to minus twice itself by H_i and to 0 by the others. Thus, the elements

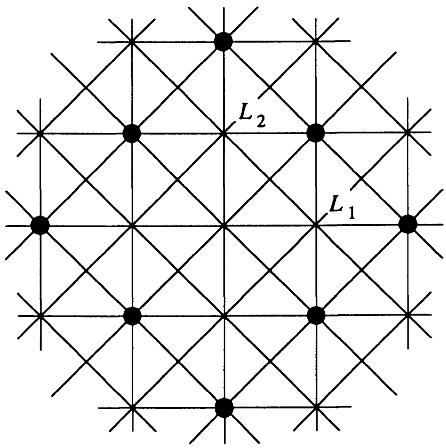
$$U_i = E_{i,n+i}$$

and

$$V_i = E_{n+i,i}$$

are eigenvectors with eigenvalues $2L_i$ and $-2L_i$, respectively. In sum, then, *the roots of the Lie algebra* $\mathfrak{sp}_{2n}\mathbb{C}$ *are the vectors* $\pm L_i \pm L_j \in \mathfrak{h}^*$.

In the first case $n = 1$, of course we just get the root diagram of $\mathfrak{sl}_2\mathbb{C}$, which is the same algebra as $\mathfrak{sp}_2\mathbb{C}$. In case $n = 2$, we have the diagram



(16.2)

As in the case of the special linear Lie algebras, probably the easiest way to determine the Killing form on $\mathfrak{sp}_{2n}\mathbb{C}$ (at least up to scalars) is to use its invariance under the automorphisms of $\mathfrak{sp}_{2n}\mathbb{C}$ preserving $\mathfrak{h}$. For example, we have the automorphisms of $\mathfrak{sp}_{2n}\mathbb{C}$ induced by permutations of the basis vectors e_i of V: for any permutation σ of $\{1, 2, \dots, n\}$ we can define an automorphism of V preserving Q by sending e_i to $e_{\sigma(i)}$ and e_{n+i} to $e_{n+\sigma(i)}$, and this induces an automorphism of $\mathfrak{sp}_{2n}\mathbb{C}$ preserving $\mathfrak{h}$ and carrying H_i to $H_{\sigma(i)}$. Also, for any i we can define an involution of V—and thereby of $\mathfrak{sp}_{2n}\mathbb{C}$—by sending e_i to e_{n+i}, e_{n+i} to $-e_i$, and all the other basis vectors to themselves; this will have the effect of sending H_i to $-H_i$ and preserving all the other H_j. Now, the Killing form on $\mathfrak{h}$ must be invariant under these automorphisms; from the first batch it follows that for some pair of constants α and β we must have

$$B(H_i, H_i) = \alpha$$

and

$$B(H_i, H_j) = \beta \quad \text{for } i \neq j;$$

from the second batch it follows that, in fact, $\beta = 0$. Thus, B is just a multiple of the standard quadratic form $B(H_i, H_j) = \delta_{i,j}$, and the dual form correspondingly a multiple of $B(L_i, L_j) = \delta_{i,j}$; so that the angles in the diagram above are correct.

Also as in the case of $\mathfrak{sl}_n\mathbb{C}$, one can also compute the Killing form directly from the definition: $B(H, H') = \sum \alpha(H)\alpha(H')$, the sum over all roots α. For $H = \sum a_i H_i$ and $H' = \sum b_i H_i$, this gives $B(H, H')$ as a sum

$$\sum_{i \neq j} (a_i + a_j)(b_i + b_j) + 2\sum_i (2a_i)(2bi) + \sum_{i \neq j} (a_i - a_j)(b_i - b_j)$$

which simplifies to

$$B(H, H') = (4n + 4)(\textstyle\sum a_i b_i). \tag{16.3}$$

Our next job is to locate the distinguished copies $\mathfrak{s}_\alpha$ of $\mathfrak{sl}_2\mathbb{C}$, and the corresponding elements $H_\alpha \in \mathfrak{h}$. This is completely straightforward. We start with the eigenvalues $L_i - L_j$ and $L_j - L_i$ corresponding to the elements $X_{i,j}$ and $X_{j,i}$; we have

$$\begin{aligned}
[X_{i,j}, X_{j,i}] &= [E_{i,j} - E_{n+j,n+i}, E_{j,i} - E_{n+i,n+j}] \\
&= [E_{i,j}, E_{j,i}] + [E_{n+j,n+i}, E_{n+i,n+j}] \\
&= E_{i,i} - E_{j,j} + E_{n+j,n+j} - E_{n+i,n+i} \\
&= H_i - H_j.
\end{aligned}$$

Thus, the distinguished element $H_{L_i - L_j}$ is a multiple of $H_i - H_j$. To see what multiple, recall that $H_{L_i - L_j}$ should act on $X_{i,j}$ by multiplication by 2 and on $X_{j,i}$ by multiplication by -2; since we have

$$\begin{aligned}
\operatorname{ad}(H_i - H_j)(X_{i,j}) &= ((L_i - L_j)(H_i - H_j)) \cdot X_{i,j} \\
&= 2X_{i,j},
\end{aligned}$$

we conclude that

$$H_{L_i - L_j} = H_i - H_j.$$

Next consider the pair of opposite eigenvalues $L_i + L_j$ and $-L_i - L_j$, corresponding to the eigenvectors $Y_{i,j}$ and $Z_{i,j}$. We have

$$\begin{aligned}[Y_{i,j}, Z_{i,j}] &= [E_{i,n+j} + E_{j,n+i}, E_{n+i,j} + E_{n+j,i}] \\ &= [E_{i,n+j}, E_{n+j,i}] + [E_{j,n+i}, E_{n+i,j}] \\ &= E_{i,i} - E_{n+j,n+j} + E_{j,j} - E_{n+i,n+i} \\ &= H_i + H_j.\end{aligned}$$

We calculate then

$$\begin{aligned}\mathrm{ad}(H_i + H_j)(Y_{i,j}) &= ((L_i + L_j)(H_i + H_j)) \cdot Y_{i,j} \\ &= 2 \cdot Y_{i,j},\end{aligned}$$

so we have

$$H_{L_i + L_j} = H_i + H_j$$

and similarly

$$H_{-L_i - L_j} = -H_i - H_j.$$

Finally, we look at the pair of eigenvalues $\pm 2L_i$ coming from the eigenvectors U_i and V_i. To complete the span of U_i and V_i to a copy of $\mathfrak{sl}_2\mathbb{C}$ we add

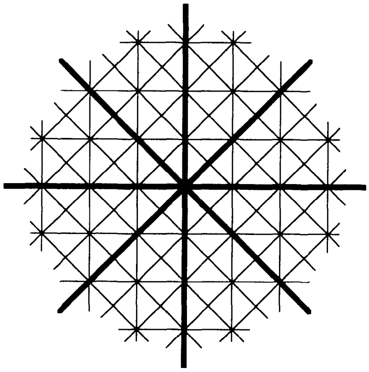
$$\begin{aligned}[U_i, V_i] &= [E_{i,n+i}, E_{n+i,i}] \\ &= E_{i,i} - E_{n+i,n+i} \\ &= H_i.\end{aligned}$$

Since

$$\begin{aligned}\mathrm{ad}(H_i)(U_i) &= (2L_i(H_i)) \cdot U_i \\ &= 2 \cdot U_i,\end{aligned}$$

we conclude that the distinguished element H_{2L_i} is H_i, and likewise $H_{-2L_i} = -H_i$. Thus, the distinguished elements $\{H_\alpha\} \subset \mathfrak{h}$ are $\{\pm H_i \pm H_j, \pm H_i\}$; in particular, the weight lattice Λ_W of linear forms on $\mathfrak{h}$ integral on all the H_α is exactly the lattice of integral linear combinations of the L_i. In Diagram (16.2), for example, this is just the lattice of intersections of the horizontal and vertical lines drawn; observe that for all n the index $[\Lambda_W : \Lambda_R]$ of the root lattice in the weight lattice is just 2.

Next we consider the group of symmetries of the weights of an arbitrary representation of $\mathfrak{sp}_{2n}\mathbb{C}$. For each root α we let W_α be the involution in $\mathfrak{h}^*$ fixing the hyperplane Ω_α given by $\langle H_\alpha, L \rangle = 0$ and acting as $-I$ on the line spanned by α; we observe in this case that, as we claimed will be true in general, the line generated by α is perpendicular to the hyperplane Ω_α, so that the involution is just a reflection in this plane. In the case $n = 2$, for example, we get the dihedral group generated by reflections around the four lines drawn through the origin:



so that the weight diagram of a representation of $\mathfrak{sp}_4\mathbb{C}$ will look like an octagon in general, or (in some cases) a square.

In general, reflection in the plane Ω_{2L_i} given by $\langle H_i, L\rangle = 0$ will simply reverse the sign of L_i while leaving the other L_j fixed; reflection in the plane $\langle H_i - H_j, L\rangle = 0$ will exchange L_i and L_j and leave the remaining L_k alone. The Weyl group $\mathfrak{W}$ acts as the full automorphism group of the lines spanned by the L_i and fits into a sequence

$$1 \to (\mathbb{Z}/2\mathbb{Z})^n \to \mathfrak{W} \to \mathfrak{S}_n \to 1.$$

Note that the sequence splits: $\mathfrak{W}$ is a semidirect product of $\mathfrak{S}_n$ and $(\mathbb{Z}/2\mathbb{Z})^n$. (This is a special case of a *wreath product*.) In particular the order of $\mathfrak{W}$ is $2^n n!$.

We can choose a positive direction as before:

$$l(\textstyle\sum a_i L_i) = c_1 a_1 + \cdots + c_n a_n, \qquad c_1 > c_2 > \cdots > c_n > 0.$$

The positive roots are then

$$R^+ = \{L_i + L_j\}_{i \le j} \cup \{L_i - L_j\}_{i<j}, \tag{16.4}$$

with primitive positive roots $\{L_i - L_{i+1}\}_{i=1,\ldots,n-1}$ and $2L_n$. The corresponding (closed) Weyl chamber is

$$\mathscr{W} = \{a_1 L_1 + a_2 L_2 + \cdots + a_n L_n \colon a_1 \ge a_2 \ge \cdots \ge a_n \ge 0\}; \tag{16.5}$$

note that the walls of this chamber—the cones

$$\{\textstyle\sum a_i L_i \colon a_1 > \cdots > a_i = a_{i+1} > \cdots > a_n > 0\}$$

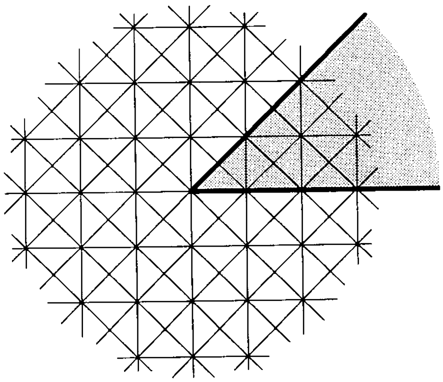
and

$$\{\textstyle\sum a_i L_i \colon a_1 > a_2 > \cdots > a_n = 0\}$$

lie in the hyperplanes $\Omega_{L_i - L_{i+1}}$ and Ω_{2L_n} perpendicular to the primitive positive or negative roots, as expected.

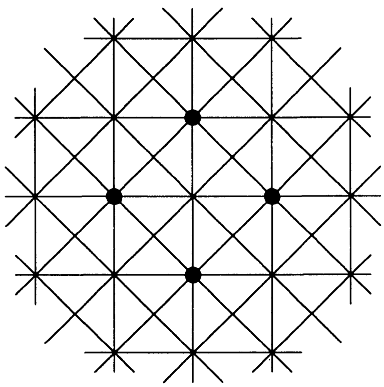
§16.2. Representations of $\mathfrak{sp}_4\mathbb{C}$

Let us consider now the representations of the algebra $\mathfrak{sp}_4\mathbb{C}$ specifically. Recall that, with the choice of Weyl chamber as above, there is a unique irreducible representation Γ_α of $\mathfrak{sp}_4\mathbb{C}$ with highest weight α for any α in the intersection of the closed Weyl chamber $\mathscr{W}$ with the weight lattice: that is, for each lattice vector in the shaded region in the diagram



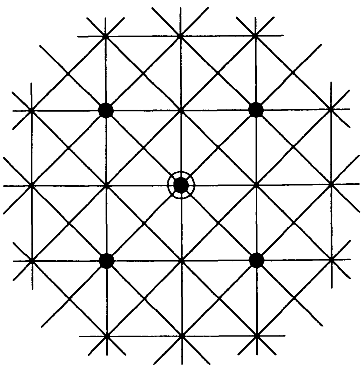
Any such highest weight vector can be written as a non-negative integral linear combination of L_1 and $L_1 + L_2$; for simplicity we will just write $\Gamma_{a,b}$ for the irreducible representation $\Gamma_{aL_1+b(L_1+L_2)}$ with highest weight $aL_1 + b(L_1 + L_2) = (a + b)L_1 + bL_2$.

To begin with, we have the standard representation as the algebra of endomorphisms of the four-dimensional vector space V; the four standard basis vectors e_1, e_2, e_3, and e_4 are eigenvectors with eigenvalues $L_1, L_2, -L_1$, and $-L_2$, respectively, so that the weight diagram of V is



V is just the representation $\Gamma_{1,0}$ in the notation above. Note that the dual of this representation is isomorphic to it, which we can see either from the symmetry of the weight diagram, or directly from the fact that the corresponding group representation preserves a bilinear form $V \times V \to \mathbb{C}$ giving an identification of V with V^*.

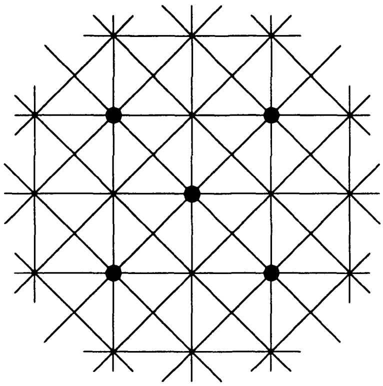
The next representation to consider is the exterior square $\wedge^2 V$. The weights of $\wedge^2 V$, the pairwise sums of distinct weights of V, are just the linear forms $\pm L_i \pm L_j$ (each appearing once) and 0 (appearing twice, as $L_1 - L_1$ and $L_2 - L_2$), so that its weight diagram looks like



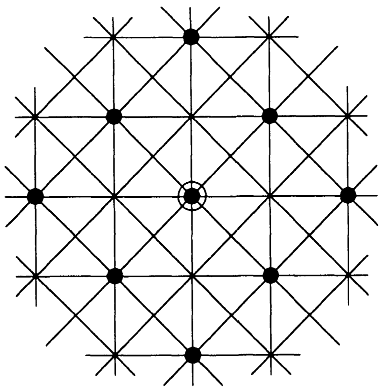
Clearly this representation is not irreducible. We can see this from the weight diagram, using Observation 14.16: there is only one way of getting to the weight space 0 from the highest weight $L_1 + L_2$ by successive applications of the primitive negative root spaces $\mathfrak{g}_{-L_1+L_2}$ (spanned by $X_{2,1} = E_{2,1} - E_{3,4}$) and $\mathfrak{g}_{-2L_2}$ (spanned by $V_2 = E_{4,2}$)—that is, by applying first V_2, which takes you to the weight space of $L_1 - L_2$, and then $X_{2,1}$—and so the dimension of the zero weight space in the irreducible representation $\Gamma_{0,1}$ with highest weight $L_1 + L_2$ must be one. Of course, we know in any event that $\wedge^2 V$ cannot be irreducible: the corresponding group action of $\mathrm{Sp}_4\mathbb{C}$ on V by definition preserves the skew form $Q \in \wedge^2 V^* \cong \wedge^2 V$. Either way, we conclude that we have a direct sum decomposition

$$\wedge^2 V = W \oplus \mathbb{C},$$

where W is the irreducible, five-dimensional representation of $\mathfrak{sp}_4\mathbb{C}$ with highest weight $L_1 + L_2$—in our notation, $\Gamma_{0,1}$—and weight diagram



Let us consider next some degree 2 tensors in V and W. To begin with, we can write down the weight diagram for the representation $\mathrm{Sym}^2 V$; the weights being just the pairwise sums of the weights of V, the diagram is

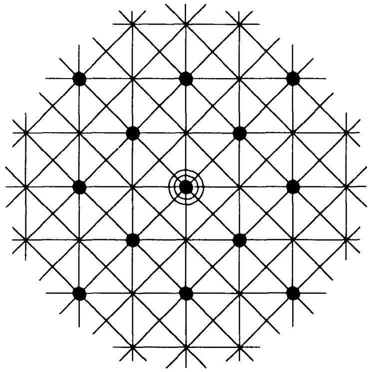


This looks like the weight diagram of the adjoint representation, and indeed that is what it is: in terms of the identification of V and V^* given by the skew form Q, the relation (16.1) defining the symplectic Lie algebra says that the subspace

$$\mathfrak{sp}_4\mathbb{C} \subset \mathrm{Hom}(V, V) = V \otimes V^* = V \otimes V$$

is just the subspace $\mathrm{Sym}^2 V \subset V \otimes V$. In particular, $\mathrm{Sym}^2 V$ is the irreducible representation $\Gamma_{2,0}$ with highest weight $2L_1$.

Next, consider the symmetric square $\mathrm{Sym}^2 W$, which has weight diagram



To see if this is irreducible we first look at the weight diagram: this time there are three ways of getting from the weight space with highest weight $2L_1 + 2L_2$ to the space of weight 0 by successively applying $X_{2,1} = E_{2,1} - E_{3,4}$ and $V_2 = E_{4,2}$, so if we want to proceed by this method we are forced to do a little calculation, which we leave as Exercise 16.7.

Alternatively, we can see directly that $\mathrm{Sym}^2 W$ decomposes: the natural map given by wedge product

$$\wedge^2 V \otimes \wedge^2 V \to \wedge^4 V = \mathbb{C}$$

is symmetric, and so factors to give a map

$$\mathrm{Sym}^2(\wedge^2 V)) \to \mathbb{C}.$$

Moreover, since this map is well defined up to scalars—in particular, it does not depend on the choice of skew form Q—it cannot contain the subspace $\mathrm{Sym}^2 W \subset \mathrm{Sym}^2(\wedge^2 V))$ in its kernel, so that it restricts to give a surjection

$$\varphi\colon \mathrm{Sym}^2 W \to \mathbb{C}.$$

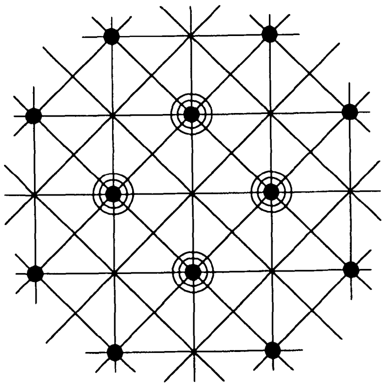
This approach would appear to leave two possibilities open: either the kernel of this map is irreducible, or it is the direct sum of an irreducible representation and a further trivial summand. In fact, however, from the principle that an irreducible representation cannot have two independent invariant bilinear forms, we see that $\mathrm{Sym}^2 W$ can contain at most one trivial summand, and so the former alternative must hold, i.e., we have

$$\mathrm{Sym}^2 W = \Gamma_{0,2} \oplus \mathbb{C}. \tag{16.6}$$

Exercise 16.7*. Prove (16.6) directly, by showing that if v is a highest weight vector, then the three vectors $X_{2,1}V_2X_{2,1}V_2v$, $X_{2,1}X_{2,1}V_2V_2v$, and $V_2X_{2,1}X_{2,1}V_2v$ span a two-dimensional subspace of the kernel of φ.

Exercise 16.8. Verify that $\wedge^2 W \cong \mathrm{Sym}^2 V$. The significance of this isomorphism will be developed further in Lecture 18.

Lastly, consider the tensor product $V \otimes W$. First, its weight diagram:



This obviously must contain the irreducible representation $\Gamma_{1,1}$ with highest weight $2L_1 + L_2$; but it cannot be irreducible, for either of two reasons. First, looking at the weight diagram, we see that $\Gamma_{1,1}$ can take on the eigenvalues $\pm L_i$ with multiplicity at most 2, so that $V \otimes W$ must contain at least one copy of the representation V. Alternatively, we have a natural map given by wedge product

$$\wedge: V \otimes \wedge^2 V \to \wedge^3 V = V^* = V;$$

and since this map does not depend on the choice of skew form Q, it must restrict to give a nonzero (and hence surjective) map

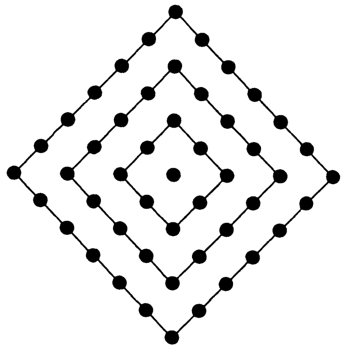
$$\varphi: V \otimes W \to V.$$

Exercise 16.9. Show that the kernel of this map is irreducible, and hence that we have

$$V \otimes W = \Gamma_{1,1} \oplus V.$$

What about more general tensors? To begin with, note that we have established the existence half of the standard existence and uniqueness theorem (14.18) in the case of $\mathfrak{sp}_4\mathbb{C}$: the irreducible representation $\Gamma_{a,b}$ may be found somewhere in the tensor product $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b W$. The question that remains is, where? In other words, we would like to be able to say how these tensor products decompose. This will be, as it was in the case of $\mathfrak{sl}_3\mathbb{C}$, nearly tantamount (modulo the combinatorics needed to count the multiplicity with which the tensor product $\mathrm{Sym}^a V \otimes \mathrm{Sym}^b W$ assumes each of its eigenvalues) to specifying the multiplicities of the irreducible representations $\Gamma_{a,b}$.

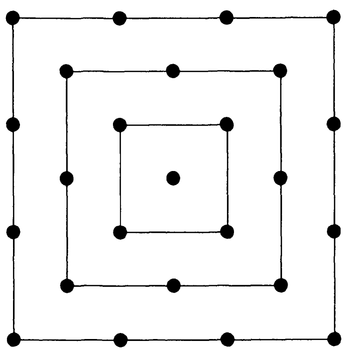
Let us start with the simplest case, namely, the representations $\mathrm{Sym}^a V$. These have weight diagram a sequence of nested diamonds D_i with vertices at $aL_1, (a-2)L_1$, etc.:



Moreover, it is not hard to calculate the multiplicities of $\mathrm{Sym}^a V$: the multiplicity on the outer diamond D_1 is one, of course; and then the multiplicities will increase by one on successive rings, so that the multiplicity along the diamond D_i will be i.

Exercise 16.10. Using the techniques of Lecture 13, show that the representations $\mathrm{Sym}^a V$ are irreducible.

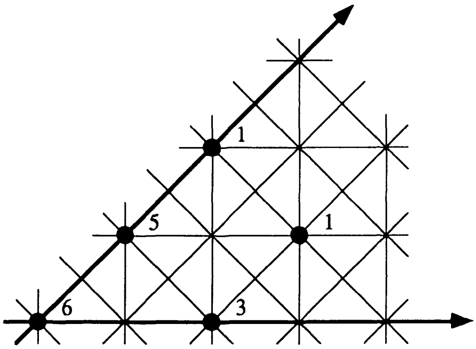
The next simplest representations, naturally enough, are the symmetric powers $\mathrm{Sym}^b W$ of W. These have eigenvalue diagrams in the shape of a sequence of squares S_i with vertices at $b(L_1 + L_2)$, $(b - 1)(L_1 + L_2)$, and so on:



Here, however, the multiplicities increase in a rather strange way: they grow quadratically, but only on every other ring. Explicitly, the multiplicity will be one on the outer two rings, then 3 on the next two rings, 6 on the next two; in general, it will be $i(i + 1)/2$ on the $(2i - 1)$st and $(2i)$th squares S_{2i-1} and S_{2i}.

Exercise 16.11. Show that contraction with the skew form $\varphi \in \mathrm{Sym}^2 W^*$ introduced in the discussion of $\mathrm{Sym}^2 W$ above determines a surjection from $\mathrm{Sym}^b W$ onto $\mathrm{Sym}^{b-2} W$, and that the kernel of this map is the irreducible representation $\Gamma_{0,b}$ with highest weight $b(L_1 + L_2)$. Show that the multiplicities of $\Gamma_{0,b}$ are i on the squares S_{2i-1} and S_{2i} described above.

We will finish by analyzing, naively and in detail, one example of a representation $\Gamma_{a,b}$ with a and b both nonzero, namely, $\Gamma_{2,1}$; one thing we may observe on the basis of this example is that there is not a similarly simple pattern to the multiplicities of the representations $\Gamma_{a,b}$ with general a and b. To carry out our analysis, we start of course with the product $\mathrm{Sym}^2 V \otimes W$. We can readily draw the weight diagram for this representation; drawing only one-eighth of the plane and indicating multiplicities by numbers, it is



We know that the representation $\mathrm{Sym}^2 V \otimes W$ contains a copy of the irreducible representation $\Gamma_{2,1}$ with highest weight $2L_1 + (L_1 + L_2)$; and we can see immediately from the diagram that it cannot equal this: for example, $\Gamma_{2,1}$ can take the weight $2L_1$ with multiplicity at most 2 (if $v \in \Gamma_{2,1}$ is its highest weight vector, the corresponding weight space $(\Gamma_{2,1})_{2L_1} \subset \Gamma_{2,1}$ will be spanned by the two vectors $X_{2,1}(Y_2(v))$ and $Y_2(X_{2,1}(v))$); since it cannot contain a copy of the representation $\Gamma_{0,2}$ (the multiplicity of the weight $2(L_1 + L_2)$ being just one) it follows that $\mathrm{Sym}^2 V \otimes W$ must contain a copy of the representation $\Gamma_{2,0} = \mathrm{Sym}^2 V$.

We can, in this way, narrow down the list of possibilities a good deal. For example, $\Gamma_{2,1}$ cannot have multiplicity just one at each of the weights $2L_1$ and $L_1 + L_2$: if it did, $\mathrm{Sym}^2 V \otimes W$ would have to contain two copies of $\mathrm{Sym}^2 V$ and a further two copies of W to make up the multiplicity at $L_1 + L_2$; but since 0 must appear as a weight of $\Gamma_{2,1}$, this would give a total multiplicity of at least 7 for the weight 0 in $\mathrm{Sym}^2 V \otimes W$. Similarly, $\Gamma_{2,1}$ cannot have multiplicity 1 at $2L_1$ and 2 at $L_1 + L_2$: we would then have two copies of $\mathrm{Sym}^2 V$ and one of W in $\mathrm{Sym}^2 V \otimes W$; and since the multiplicity of 0 in $\Gamma_{2,1}$ will in this case be at least 2 (being greater than or equal to the multiplicity of $L_1 + L_2$), this would again imply a multiplicity of at least 7 for the weight 0 in $\mathrm{Sym}^2 V \otimes W$. It follows that $\mathrm{Sym}^2 V \otimes W$ must contain exactly one copy of $\mathrm{Sym}^2 V$; and since the multiplicity of $L_1 + L_2$ in $\Gamma_{2,1}$ is at most 3, it follows that $\mathrm{Sym}^2 V \otimes W$ will contain at least one copy of $\Gamma_{0,1} = W$ as well.

Exercise 16.12. Prove, independently of the above analysis, that $\mathrm{Sym}^2 V \otimes W$ must contain a copy of $\mathrm{Sym}^2 V$ and a copy of W by looking at the map

$$\varphi\colon \mathrm{Sym}^2 V \otimes W \to V \otimes V$$

obtained by sending

$$u \cdot v \otimes (w \wedge z) \mapsto u \otimes \tilde{Q}(v \wedge w \wedge z) + v \otimes \tilde{Q}(u \wedge w \wedge z),$$

where we are identifying $\wedge^3 V$ with the dual space V^* and denoting by $\tilde{Q}\colon V^* \to V$ the isomorphism induced by the skew form Q on V. Specifically, show that the image of this map is complementary to the line spanned by the element $Q \in \wedge^2 V^* = \wedge^2 V \subset V \otimes V$.

The above leaves us with exactly two possibilities for the weights of $\Gamma_{2,1}$: we know that the multiplicity of $2L_1$ in $\Gamma_{2,1}$ is exactly 2; so either the multiplicities of $L_1 + L_2$ and 0 in $\Gamma_{2,1}$ are both 3 and we have

$$\mathrm{Sym}^2 V \otimes W = \Gamma_{2,1} \oplus \mathrm{Sym}^2 V \oplus W;$$

or the multiplicities of $L_1 + L_2$ and 0 in $\Gamma_{2,1}$ are both 2 and we have

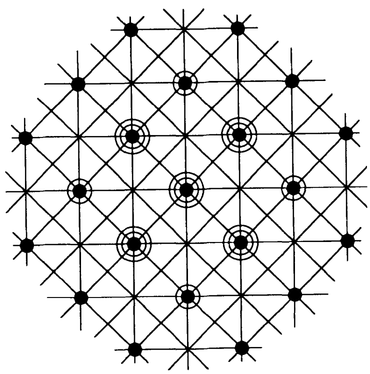
$$\mathrm{Sym}^2 V \otimes W = \Gamma_{2,1} \oplus \mathrm{Sym}^2 V \oplus W^{\oplus 2}.$$

Exercise 16.13. Show that the former of these two possibilities actually occurs, by

(a) Showing that if v is the highest weight vector in $\Gamma_{2,1} \subset \mathrm{Sym}^2 V \otimes W$, then the images $(X_{2,1})^2 Y_2(v)$, $X_{2,1} Y_2 X_{2,1}(v)$, and $Y_2(X_{2,1})^2 v$ are independent; and (redundantly)

(b) Showing that the representation $\mathrm{Sym}^2 V \otimes W$ contains only one highest weight vector of weight $L_1 + L_2$.

The weight diagram of $\Gamma_{2,1}$ is therefore



We see from all this that, in particular, the weights of the irreducible representations of $\mathfrak{sp}_4\mathbb{C}$ are not constant on the rings of their weight diagrams.

Exercise 16.14. Analyze the representation $V \otimes \mathrm{Sym}^2 W$ of $\mathfrak{sp}_4\mathbb{C}$. Find in particular the multiplicities of the representation $\Gamma_{1,2}$.

Exercise 16.15. Analyze the representation $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 W$ of $\mathfrak{sp}_4\mathbb{C}$. Find in particular the multiplicities of the representation $\Gamma_{2,2}$.

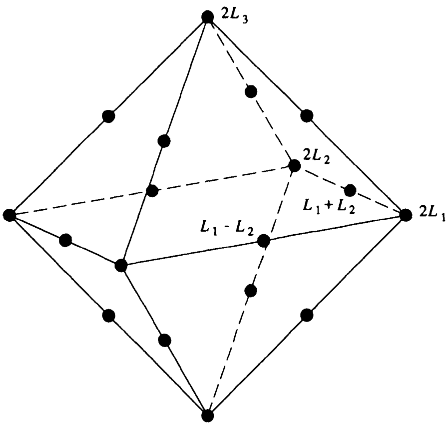
LECTURE 17

$\mathfrak{sp}_6\mathbb{C}$ and $\mathfrak{sp}_{2n}\mathbb{C}$

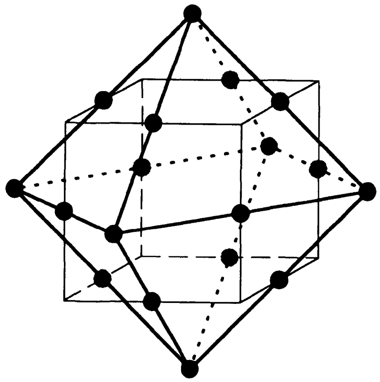
In the first two sections of this lecture we complete our classification of the representations of the symplectic Lie algebras: we describe in detail the example of $\mathfrak{sp}_6\mathbb{C}$, then sketch the representation theory of symplectic Lie algebras in general, in particular proving the existence part of Theorem 14.18 for $\mathfrak{sp}_{2n}\mathbb{C}$. In the final section we describe an analog for the symplectic algebras of the construction given in §15.3 of the irreducible representations of the special linear algebras via Weyl's construction, though we postpone giving analogous formulas for the decomposition of tensor products of irreducible representations. Sections 17.1 and 17.2 are completely elementary, given the by now standard multilinear algebra of Appendix B. Section 17.3, like §15.3, requires familiarity with the contents of Lecture 6 and Appendix A; but, like that section, it can be skipped without affecting most of the rest of the book.

§17.1. Representations of $\mathfrak{sp}_6\mathbb{C}$

As we have seen, the Cartan algebra $\mathfrak{h}$ of $\mathfrak{sp}_6\mathbb{C}$ is three-dimensional, with the linear functionals L_1, L_2, and L_3 forming an orthonormal basis in terms of the Killing form; and the roots of $\mathfrak{sp}_6\mathbb{C}$ are then the 18 vectors $\pm L_i \pm L_j$. We can draw this in terms of a "reference cube" in $\mathfrak{h}^*$ with faces centered at the points $\pm L_i$; the vectors $\pm L_i \pm L_j$ with $i \neq j$ are then the midpoints of edges of this reference cube and the vectors $\pm 2L_i$ the midpoints of the faces of a cube twice as large. Alternatively, we can draw a reference octahedron with vertices at the vectors $\pm 2L_i$; the roots $\pm L_i \pm L_j$ with $i \neq j$ will then be the midpoints of the edges of this octahedron:

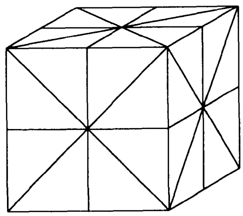


or, if we include the reference cube as well, as

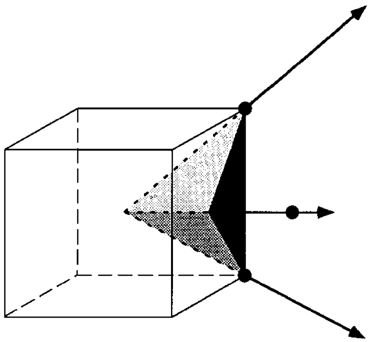


(17.1)

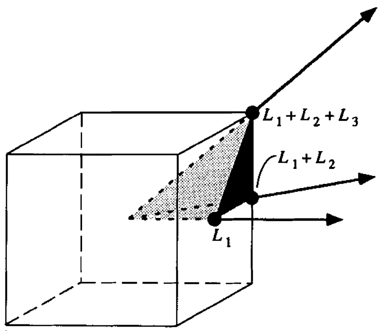
This last diagram suggests a comparison with the root diagram of $\mathfrak{sl}_4\mathbb{C}$; in fact the 12 roots of $\mathfrak{sp}_6\mathbb{C}$ of the form $\pm L_i \pm L_j$ for $i \neq j$ are congruent to the 12 roots of $\mathfrak{sl}_4\mathbb{C}$. In particular, the Weyl group of $\mathfrak{sp}_6\mathbb{C}$ will be generated by the Weyl group of $\mathfrak{sl}_4\mathbb{C}$, plus any of the additional three reflections in the planes perpendicular to the L_i (i.e., the planes parallel to the faces of the reference cube in the root diagram of either Lie algebra). We can indicate the planes perpendicular to the roots of $\mathfrak{sp}_6\mathbb{C}$ by drawing where they cross the visible part of the reference cube:



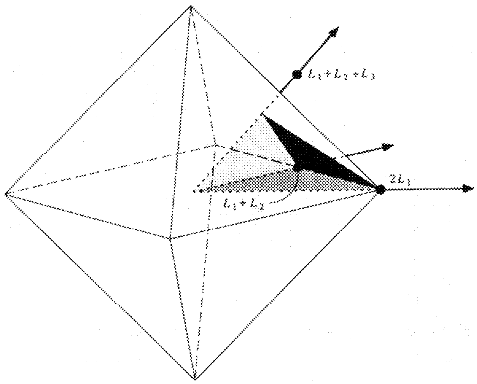
We see from this that the effect of the additional reflections in the Weyl group of $\mathfrak{sp}_6\mathbb{C}$ on the Weyl chamber of $\mathfrak{sl}_4\mathbb{C}$ is simply to cut it in half; whereas the Weyl chamber of $\mathfrak{sl}_4\mathbb{C}$ looked like



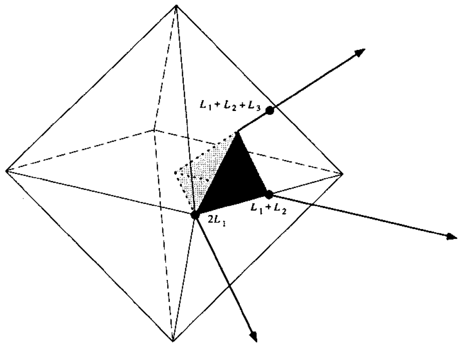
the Weyl chamber of $\mathfrak{sp}_6\mathbb{C}$ will look like just the upper half of this region:



In terms of the reference octahedron, this is the cone over one part of the barycentric subdivision of a face:



or, if we rotate 90° around the vertical axis in an attempt to make the picture clearer,



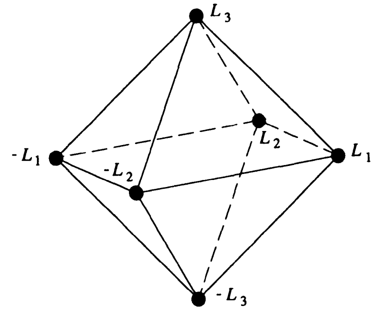
(17.2)

We should remark before proceeding that the comparison between the root systems of the special linear algebra $\mathfrak{sl}_4\mathbb{C}$ and the symplectic algebra $\mathfrak{sp}_6\mathbb{C}$ is peculiar to this case; in general, the root systems of $\mathfrak{sl}_{n+1}\mathbb{C}$ and $\mathfrak{sp}_{2n}\mathbb{C}$ will bear no such similarity.

As we saw in the preceding lecture, the weight lattice of $\mathfrak{sp}_6\mathbb{C}$ consists simply of the integral linear combinations of the weights L_i. In particular, the intersection of the weight lattice with the closed Weyl chamber chosen above will consist exactly of integral linear combinations $a_1L_1 + a_2L_2 + a_3L_3$ with $a_1 \geq a_2 \geq a_3 \geq 0$. By our general existence and uniqueness theorem, then, for every triple (a, b, c) of non-negative integers there will exist a unique irreducible representation of $\mathfrak{sp}_6\mathbb{C}$ with highest weight $aL_1 + b(L_1 + L_2) +$

$c(L_1 + L_2 + L_3) = (a + b + c)L_1 + (b + c)L_2 + cL_3$; we will denote this representation by $\Gamma_{a,b,c}$ and will demonstrate its existence in the following.

We start by considering the standard representation of $\mathfrak{sp}_6\mathbb{C}$ on $V = \mathbb{C}^6$. The eigenvectors of the action of $\mathfrak{h}$ on V are just the standard basis vectors e_i, and these have eigenvalues $\pm L_i$, so that the weight diagram of V looks like the midpoints of the faces of the reference cube (or the vertices of an octahedron one-half the size of the reference octahedron):



In particular, V is the representation $\Gamma_{1,0,0}$.

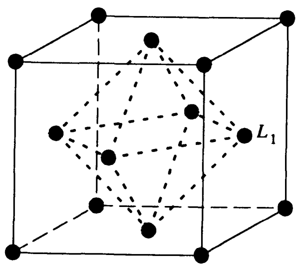
Since we are going to want to find a representation with highest weight $L_1 + L_2$, the natural thing to look at next is the second exterior power $\wedge^2 V$ of the standard representation. This will have weights the pairwise sum of distinct weights of V, or in other words the 12 weights $\pm L_i \pm L_j$ with $i \neq j$, and the weight 0 taken three times. This is not irreducible: by definition the action of $\mathfrak{sp}_6\mathbb{C}$ on the standard representation preserves a skew form, so that the representation on $\wedge^2 V$ will have a trivial summand. On the other hand, the skew form on V preserved by $\mathfrak{sp}_6\mathbb{C}$, and hence that trivial summand of $\wedge^2 V$, is unique; and since all the nonzero weights of $\wedge^2 V$ occur with multiplicity 1 and are conjugate under the Weyl group, it follows that the complement W of the trivial representation in $\wedge^2 V$ is irreducible. So $W = \Gamma_{0,1,0}$.

As in previous examples, we can also see that $\wedge^2 V$ is not irreducible by using the fact (Observation 14.16) that the irreducible representation $\Gamma_{0,1,0}$ with highest weight $L_1 + L_2$ will be generated by applying to a single highest weight vector v the root spaces $\mathfrak{g}_{L_2-L_1}$, $\mathfrak{g}_{L_3-L_2}$, and $\mathfrak{g}_{-2L_3}$ corresponding to primitive negative roots. We can then verify that in the irreducible representation W with highest weight $L_1 + L_2$, there are only three ways of going from the highest weight space to the zero weight space by successive application of these roots spaces: we can go

$$\begin{tikzcd}
L_1 + L_2 \arrow[r] & L_1 + L_3 \arrow[r] \arrow[dr] & L_1 - L_3 \arrow[r] \arrow[dr] & L_1 - L_2 \arrow[dr] & \\
 & & L_2 + L_3 \arrow[r] & L_2 - L_3 \arrow[r] & 0
\end{tikzcd}$$

Exercise 17.3. Verify this, and also verify that the lower two routes to the zero-weight space in $\wedge^2 V$ yield the same nonzero vector, and that the upper route yields an independent element of $\wedge^2 V$, so that 0 does indeed occur with multiplicity 2 as a weight of $\Gamma_{0,1,0}$.

To continue, we look next at the third exterior power $\wedge^3 V$ of the standard representation; we know that this will contain a copy of the irreducible representation $\Gamma_{0,0,1}$ with highest weight $L_1 + L_2 + L_3$. The weights of $\wedge^3 V$ are of two kinds: we have the eight sums $\pm L_1 \pm L_2 \pm L_3$, corresponding to the vertices of the reference cube and each occurring once; and we have the weights $\pm L_i$ each occurring twice (as $\pm L_i + L_j - L_j$ and $\pm L_i + L_k - L_k$). The weight diagram thus looks like the vertices of the reference cube together with the midpoints of its faces:



Now, the weights $\pm L_i$ must occur in the representation $\Gamma_{0,0,1}$ with highest weight $L_1 + L_2 + L_3$, since they are congruent to $L_1 + L_2 + L_3$ modulo the root lattice and lie in the convex hull of the translates of $L_1 + L_2 + L_3$ under the Weyl group (that is, they lie in the closed reference cube). But they cannot occur with multiplicity greater than 1: for example, the only way to get from the point $L_1 + L_2 + L_3$ to the point L_1 by translations by the basic vectors $L_2 - L_1, L_3 - L_2$, and $-2L_3$ pictured in Diagram (17.1) above (while staying inside the reference cube) is by translation by $-2L_3$ first, and then by $L_3 - L_2$. it follows that the multiplicities of the weights $\pm L_i$ in $\Gamma_{0,0,1}$ are 1. On the other hand, we have a natural map

$$\wedge^3 V \to V$$

obtained by contracting with the element of $\wedge^2 V^*$ preserved by the action of $\mathfrak{sp}_6\mathbb{C}$, and the kernel of this map, which must contain the representation $\Gamma_{0,0,1}$, will have exactly these weights. The kernel of φ is thus the irreducible representation with highest weight $L_1 + L_2 + L_3$; we will call this representation U for now.

At this point, we have established the existence theorem for representations of $\mathfrak{sp}_6\mathbb{C}$: the irreducible representation $\Gamma_{a,b,c}$ with highest weight $(a + b + c)L_1 + (a + b)L_2 + cL_3$ will occur inside the representation

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b W \otimes \mathrm{Sym}^c U.$$

For example, suppose we want to find the irreducible representation $\Gamma_{1,1,0}$ with highest weight $2L_1 + L_2$. The weights of this representation will be the 24 weights $\pm 2L_i \pm L_j$, each taken with multiplicity 1; the 8 weights $\pm L_1 \pm L_2 \pm L_3$, taken with a multiplicity we do not a priori know (but that the reader can verify must be either 1 or 2), and the weights $\pm L_i$ taken with some other multiplicity. At the same time, the representation $V \otimes W$, which contains $\Gamma_{1,1,0}$, will take on these weights, with multiplicities 1, 3, and 6, respectively. In particular, it follows that $V \otimes W$ will contain a copy of the irreducible representation U with highest weight $L_1 + L_2 + L_3$ as well; alternatively, we can see this directly by observing that the wedge product map

$$V \otimes \wedge^2 V \to \wedge^3 V$$

factors to give a map

$$V \otimes W \to U$$

and that $\Gamma_{1,1,0}$ must lie in the kernel of this map. To say more about the location of $\Gamma_{1,1,0}$ inside $V \otimes W$, and its exact weights, would require either explicit calculation or something like the Weyl character formula. We will see in Lecture 24 how the latter can be used to solve the problem; for the time being we leave this as

Exercise 17.4. Verify by direct calculation that the multiplicities of the weights of $\Gamma_{1,1,0}$ are 1, 2, and 5, and hence that the kernel of the map φ above is exactly the representation $\Gamma_{1,1,0}$.

§17.2. Representations of $\mathfrak{sp}_{2n}\mathbb{C}$ in General

The general picture for representations of the symplectic Lie algebras offers no further surprises. As we have seen, the weight lattice consists simply of integral linear combinations of the L_i. And our typical Weyl chamber is a cone over a simplex in n-space, with edges the rays defined by

$$a_1 = a_2 = \cdots = a_i > a_{i+1} = \cdots = a_n = 0.$$

The primitive lattice element on the ith ray is the weight $\omega_i = L_1 + \cdots + L_i$, and we may observe that, similarly to the case of the special linear Lie algebras, these n fundamental weights generate as a semigroup the intersection of the closed Weyl chamber with the lattice. Thus, our basic existence and uniqueness theorem asserts that for an arbitrary n-tuple of natural numbers $(a_1, \ldots, a_n) \in \mathbb{N}^n$ there will be a unique irreducible representation with highest weight

$$\begin{aligned} &a_1\omega_1 + a_2\omega_2 + \cdots + a_n\omega_n \\ &\qquad = (a_1 + \cdots + a_n)L_1 + (a_2 + \cdots + a_n)L_2 + \cdots + a_nL_n. \end{aligned}$$

As before, we denote this by $\Gamma_{a_1,\ldots,a_n}$:

$$\Gamma_{a_1,\ldots,a_n} = \Gamma_{a_1L_1+a_2(L_1+L_2)+\cdots+a_n(L_1+\cdots+L_n)}.$$

These exhaust all irreducible representations of $\mathfrak{sp}_{2n}\mathbb{C}$.

We can find the irreducible representation $V^{(k)} = \Gamma_{0,\ldots,1,\ldots,0}$ with highest weight $L_1 + \cdots + L_k$ easily enough. Clearly, it will be contained in the kth exterior power $\wedge^k V$ of the standard representation. Moreover, we have a natural contraction map

$$\varphi_k\colon \wedge^k V \to \wedge^{k-2} V$$

defined by

$$\varphi_k(v_1 \wedge \cdots \wedge v_k) = \sum_{i<j} Q(v_i, v_j)(-1)^{i+j-1} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k$$

(see §B.3 of Appendix B for an intrinsic definition and explanation). Since the representation $\wedge^{k-2}V$ does not have the weight $L_1 + \cdots + L_k$, the irreducible representation with this highest weight will have to be contained in the kernel of this map. We claim now that conversely

Theorem 17.5. *For $1 \leq k \leq n$, the kernel of the map φ_k is exactly the irreducible representation $V^{(k)} = \Gamma_{0,\ldots,0,1,0,\ldots,0}$ with highest weight $L_1 + \cdots + L_k$.*

PROOF. Clearly, it is enough to show that the kernel of φ_k is an irreducible representation of $\mathfrak{sp}_{2n}\mathbb{C}$. We will do this by restricting to a subalgebra of $\mathfrak{sp}_{2n}\mathbb{C}$ isomorphic to $\mathfrak{sl}_n\mathbb{C}$, and using what we have learned about representations of $\mathfrak{sl}_n\mathbb{C}$.

To describe this copy of $\mathfrak{sl}_n\mathbb{C}$ inside $\mathfrak{sp}_{2n}\mathbb{C}$, consider the subgroup $G \subset \mathrm{Sp}_{2n}\mathbb{C}$ of transformations of the space $V = \mathbb{C}^{2n}$ preserving the skew form Q introduced in Lecture 16 and preserving as well the decomposition $V = \mathbb{C}\{e_1, \ldots, e_n\} \oplus \mathbb{C}\{e_{n+1}, \ldots, e_{2n}\}$. These can act arbitrarily on the first factor, as long as they do the opposite on the second; in coordinates, they are the matrices

$$G = \left\{\begin{pmatrix} X & 0 \\ 0 & {}^tX^{-1} \end{pmatrix}, X \in \mathrm{GL}_n\mathbb{C}\right\}.$$

We have, correspondingly, a subalgebra

$$\mathfrak{s} = \left\{\begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix}, A \in \mathfrak{sl}_n\mathbb{C}\right\} \subset \mathfrak{sp}_{2n}\mathbb{C}$$

isomorphic to $\mathfrak{sl}_n\mathbb{C}$.

Now, denote by W the standard representation of $\mathfrak{sl}_n\mathbb{C}$. The restriction of the representation V of $\mathfrak{sp}_{2n}\mathbb{C}$ to the subalgebra $\mathfrak{s}$ then splits

$$V = W \oplus W^*$$

into a direct sum of W and its dual; and we have, correspondingly,

$$\wedge^k V = \bigoplus_{a+b=k} (\wedge^a W \otimes \wedge^b W^*).$$

How does the tensor product $\wedge^a W \otimes \wedge^b W^*$ decompose as a representation of $\mathfrak{sl}_n\mathbb{C}$? We know the answer to this from the discussion in Lecture 15 (see Exercise 15.30): we have contraction maps

$$\Psi_{a,b}: \wedge^a W \otimes \wedge^b W^* \to \wedge^{a-1} W \otimes \wedge^{b-1} W^*;$$

and the kernel of $\Psi_{a,b}$ is the irreducible representation $W^{(a,b)} = \Gamma_{0,\dots,0,1,0,\dots,0,1,0,\dots}$ with (if, say, $a \le n-b$) highest weight $2L_1 + \cdots + 2L_a + L_{a+1} + \cdots + L_{n-b}$. The restriction of $\wedge^k V$ to $\mathfrak{s}$ is thus given by

$$\wedge^k V = \bigoplus_{\substack{a+b\le k\\ a+b\equiv k(2)}} W^{(a,b)}$$

and by the same token,

$$\operatorname{Ker}(\varphi_k) = \bigoplus_{a+b=k} W^{(a,b)}.$$

Note that the actual highest weight factor in the summand $W^{(a,b)} \subset \operatorname{Ker}(\varphi_k) \subset \wedge^k V$ is the vector

$$\begin{aligned} w^{(a,b)} &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \\ &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-k+a+1} \wedge \cdots \wedge e_{2n}. \end{aligned}$$

Exercise 17.6. Show that more generally the highest weight vector in any summand $W^{(a,b)} \subset \wedge^k V$ is the vector

$$\begin{aligned} w^{(a,b)} &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-k+a+1} \wedge \cdots \wedge e_{2n} \wedge Q^{(k-a-b)/2} \\ &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-k+a+1} \wedge \cdots \wedge e_{2n} \wedge \left(\sum (e_i \wedge e_{n+i})\right)^{(k-a-b)/2}. \end{aligned}$$

By the above, any subspace of $\operatorname{Ker}(\varphi_k)$ invariant under $\mathfrak{sp}_{2n}\mathbb{C}$ must be a direct sum, over a subset of pairs (a, b) with $a + b = k$, of subspaces $W^{(a,b)}$. But now (supposing for the moment that $k < n$) we observe that the element

$$Z_{a,n-b} = E_{2n-b,a} + E_{n+a,n-b} \in \mathfrak{sp}_{2n}\mathbb{C}$$

carries the vector $w^{(a,b)}$ into $w^{(a-1,b+1)}$ and, likewise,

$$Y_{a+1,n-b+1} = E_{a+1,2n-b+1} + E_{n-b+1,n+a+1} \in \mathfrak{sp}_{2n}\mathbb{C}$$

carries $w^{(a,b)}$ to $w^{(a+1,b-1)}$. In case $a + b = k = n$, we see similarly that

$$V_a = E_{n+a,a} \in \mathfrak{sp}_{2n}\mathbb{C}$$

carries the vector $w^{(a,b)}$ into $w^{(a-1,b+1)}$, and

$$U_{a+1} = E_{a+1,n+a+1} \in \mathfrak{sp}_{2n}\mathbb{C}$$

carries $w^{(a,b)}$ to $w^{(a+1,b-1)}$. Thus, any representation of $\mathfrak{sp}_{2n}\mathbb{C}$ contained in $\operatorname{Ker}(\varphi_k)$ and containing any one of the factors $W^{(a,b)}$ will contain them all, and we are done. $\square$

Exercise 17.7. Another way to conclude this proof would be to remark that, inasmuch as all the $w^{(a,b)}$ above are eigenvectors of different weights, any highest weight vector for the action of $\mathfrak{sp}_{2n}\mathbb{C}$ on $\ker(\varphi_k) \subset \wedge^k V$ would have to be (up to scalars) one of the $w^{(a,b)}$. It would thus be sufficient to find, for each (a, b) with $a + b = k$ other than $(a, b) = (k, 0)$, a positive root α such that $\mathfrak{g}_\alpha(w^{(a,b)}) \neq 0$. Do this.

Note that, having found the irreducible representations $V^{(k)} = \Gamma_{0,\ldots,1,\ldots,0}$ with highest weight $L_1 + \cdots + L_k$, any other representation of $\mathfrak{sp}_{2n}\mathbb{C}$ will occur in a tensor product of these; specifically, the irreducible representation $\Gamma_{a_1,\ldots,a_n}$ with highest weight $a_1 L_1 + \cdots + a_n(L_1 + \cdots + L_n)$ will occur in the product $\mathrm{Sym}^{a_1} V \otimes \mathrm{Sym}^{a_2} V^{(2)} \otimes \cdots \otimes \mathrm{Sym}^{a_n} V^{(n)}$.

One further remark is that there exist geometric interpretations of the action of $\mathfrak{sl}_{2n}\mathbb{C}$ on the fundamental representations $V^{(k)}$. We have said before that the group $\mathrm{PSp}_{2n}\mathbb{C}$ may be characterized as the subgroup of $\mathrm{PGL}_{2n}\mathbb{C}$ carrying isotropic subspaces of V into isotropic subspaces. At the same time, $\mathrm{PGL}_{2n}\mathbb{C}$ acts on the projective space $\mathbb{P}(\wedge^k V)$ as the connected component of the identity in the group of motions of this space carrying the Grassmannian $G = G(k, V) \subset \mathbb{P}(\wedge^k V)$ into itself. Now, the subset $G_L \subset G$ of k-dimensional isotropic subspaces of V is exactly the intersection of the Grassmannian G with the subspace $\mathbb{P}(V^{(k)})$ associated to the kernel of the map φ above; so that $\mathrm{PSp}_{2n}\mathbb{C}$ will act on $\mathbb{P}(V^{(k)})$ carrying G_L into itself and indeed when $1 < k \leq n$ may be characterized as the connected component of the identity in the group of motions of $\mathbb{P}(V^{(k)})$ preserving the variety G_L.

Exercise 17.8. Show that if $k > n$ the contraction φ_k is injective.

§17.3. Weyl's Construction for Symplectic Groups

We have just seen how the basic representations for $\mathfrak{sp}_{2n}\mathbb{C}$ can be obtained by taking certain basic representations of the larger Lie algebra $\mathfrak{sl}_{2n}\mathbb{C}$—in this case, $\wedge^k V$ for $k \leq n$—and intersecting with the kernel of a contraction constructed from the symplectic form. In fact, *all* the representations of the symplectic Lie algebras can be given a similar conrete realization, by intersecting certain of the irreducible representations of $\mathfrak{sl}_{2n}\mathbb{C}$ with the intersections of the kernels of all such contractions.

Recall from Lectures 6 and 15 that the irreducible representations of $\mathfrak{sl}_{2n}\mathbb{C}$ are given by Schur functors $\mathbb{S}_\lambda V$, where $\lambda = (\lambda_1 \geq \cdots \geq \lambda_{2n} \geq 0)$ is a partition of some integer $d = \sum \lambda_i$, and $V = \mathbb{C}^{2n}$. This representation is realized as the image of a corresponding Young symmetrizer c_λ acting on the d-fold tensor product space $V^{\otimes d}$. For each pair $I = \{p < q\}$ of integers between 1 and d, the symplectic form Q determines a contraction

$$\begin{aligned} \Phi_I\colon V^{\otimes d} &\to V^{\otimes(d-2)}, \\ v_1 \otimes \cdots \otimes v_d &\mapsto Q(v_p, v_q) v_1 \otimes \cdots \otimes \hat{v}_p \otimes \cdots \otimes \hat{v}_q \otimes \cdots \otimes v_d. \end{aligned} \tag{17.9}$$

Let $V^{\langle d\rangle} \subset V^{\otimes d}$ denote the intersection of the kernels of all these contractions. These subspaces is mapped to itself by permutations, so $V^{\langle d\rangle}$ is a subrepresentation of $V^{\otimes d}$ as a representation of the symmetric group $\mathfrak{S}_d$. Now let[1]

$$\mathbb{S}_{\langle\lambda\rangle}V = V^{\langle d\rangle} \cap \mathbb{S}_\lambda V. \tag{17.10}$$

This space is a representation of the symplectic group $\mathrm{Sp}_{2n}\mathbb{C}$ of Q, since $V^{\langle d\rangle}$ and $\mathbb{S}_\lambda(V)$ are subrepresentations of $V^{\otimes d}$ over $\mathrm{Sp}_{2n}\mathbb{C}$.

Theorem 17.11. *The space $\mathbb{S}_{\langle\lambda\rangle}(V)$ is nonzero if and only if the Young diagram of λ has at most n rows, i.e., $\lambda_{n+1} = 0$. In this case, $\mathbb{S}_{\langle\lambda\rangle}(V)$ is the irreducible representation of $\mathfrak{sp}_{2n}\mathbb{C}$ with highest weight $\lambda_1 L_1 + \cdots + \lambda_n L_n$.*

In other words, for an n-tuple $(a_1, \ldots, a_n)$ of non-negative integers

$$\Gamma_{a_1,\ldots,a_n} = \mathbb{S}_{\langle\lambda\rangle}V,$$

where λ is the partition $(a_1 + a_2 + \cdots + a_n, a_2 + \cdots + a_n, \ldots, a_n)$.

The proof follows the pattern for the general linear group given in §6.2, but we will have to call on a basic result from invariant theory in place of the simple Lemma 6.23. We first show how to find a complement to $V^{\langle d\rangle}$ in $V^{\otimes d}$. For example, if $d = 2$, then

$$V^{\otimes 2} = V^{\langle 2\rangle} \oplus \mathbb{C}\cdot\psi,$$

where ψ is the element of $V \otimes V$ corresponding to the quadratic form Q. In terms of our canonical basis, $\psi = \sum(e_i \otimes e_{n+i} - e_{n+i} \otimes e_i)$. In general, for any $I = \{p < q\}$ define

$$\Psi_I\colon V^{\otimes(d-2)} \to V^{\otimes d}$$

by inserting ψ in the p, q factors. Note that $\Phi_I \circ \Psi_I$ is multiplication by $2n = \dim V$ on $V^{\otimes(d-2)}$. We claim that

$$V^{\otimes d} = V^{\langle d\rangle} \oplus \textstyle\sum_I \Psi_I(V^{\otimes(d-2)}). \tag{17.12}$$

To prove this, put the standard Hermitian metric $(\ ,\ )$ on $V = \mathbb{C}^{2n}$, using the given e_i as a basis, so that $(ae_i, be_j) = \delta_{ij}\bar{a}b$. This extends to give a Hermitian metric on each $V^{\otimes d}$. We claim that the displayed equation is a perpendicular direct sum. This follows from the following exercise.

Exercise 17.13. (i) Verify that for $v, w \in V$, $(\psi, v \otimes w) = Q(v, w)$.
(ii) Use (i) to show that $\mathrm{Ker}(\Phi_I) = \mathrm{Im}(\Psi_I)^\perp$ for each I.

Now define $F_r^d \subset V^{\otimes d}$ to be the intersection of the kernels of all r-fold contractions $\Phi_{I_1} \circ \cdots \circ \Phi_{I_r}$, and set

$$V_{d-2r}^{\langle d\rangle} = \sum \Psi_{I_1} \circ \cdots \circ \Psi_{I_r}(V^{\otimes\langle d-2r\rangle}). \tag{17.14}$$

[1] This follows a classical notation of using $\langle\ \ \rangle$ for the symplectic group and $[\ \ ]$ for the orthogonal group (although we have omitted the corresponding notation $\{\ \ \}$ for the general linear group).

Lemma 17.15. *The tensor power $V^{\otimes d}$ decomposes into a direct sum*

$$V^{\otimes d} = V^{\langle d \rangle} \oplus V_{d-2}^{\langle d \rangle} \oplus V_{d-4}^{\langle d \rangle} \oplus \cdots \oplus V_{d-2p}^{\langle d \rangle},$$

with $p = [d/2]$, and, for all $r \geq 1$,

$$F_r^d = V^{\langle d \rangle} \oplus V_{d-2}^{\langle d \rangle} \oplus \cdots \oplus V_{d-2r+2}^{\langle d \rangle}.$$

Exercise 17.16. (i) Show as in the preceding exercise that there is a perpendicular decomposition

$$V^{\otimes d} = F_r^d \oplus \sum \Psi_{I_1} \circ \cdots \circ \Psi_{I_r}(V^{\otimes(d-2r)}).$$

(ii) Verify that $\Psi_I(F_p^{d-2}) \subset F_{p+1}^d$.

(iii) Show by induction that $V^{\otimes d}$ is the sum of the spaces $V_{d-2r}^{\langle d \rangle}$.

(iv) Finish the proof of the lemma, using (i) and (ii) to deduce that both sums are orthogonal splittings. □

All the subspaces in these splittings are invariant by the action of the symplectic group $\mathrm{Sp}_{2n}\mathbb{C}$, as well as the action of the symmetric group $\mathfrak{S}_d$. In particular, we see that

$$\mathbb{S}_{\langle\lambda\rangle}V = V^{\langle d \rangle} \cdot c_\lambda = \mathrm{Im}(c_\lambda\colon V^{\langle d \rangle} \to V^{\langle d \rangle}). \tag{17.17}$$

Exercise 17.18*. (i) Show that if $s > n$, then $\bigwedge^s V \otimes V^{\otimes(d-s)}$ is contained in $\sum_I \Psi_I(V^{\otimes(d-2)})$, and deduce that $\mathbb{S}_{\langle\lambda\rangle}(V) = 0$ if λ_{n+1} is not 0.

(ii) Show that $\mathbb{S}_{\langle\lambda\rangle}(V)$ is not zero if $\lambda_{n+1} = 0$.

For any pair of integers I from $\{1, \ldots, d\}$, define

$$\vartheta_I = \Psi_I \circ \Phi_I\colon V^{\otimes d} \to V^{\otimes d}.$$

From what we have seen, $V^{\langle d \rangle}$ is the intersection of the kernels of all these endomorphisms. Note that the endomorphism of $V^{\otimes d}$ determined by any symplectic automorphism of V not only commutes with all permutations of the factors $\mathfrak{S}_d$ but also commutes with the operators ϑ_I. We need a fact which is proved in Appendix F.2:

Invariant Theory Fact 17.19. *Any endomorphism of $V^{\otimes d}$ that commutes with all permutations in $\mathfrak{S}_d$ and all the operators ϑ_I is a finite $\mathbb{C}$-linear combination of operators of the form $A \otimes \cdots \otimes A$, for $A \in \mathrm{Sp}_{2n}\mathbb{C}$.*

Now let B be the algebra of all endomorphisms of the space $V^{\langle d \rangle}$ that are $\mathbb{C}$-linear combinations of operators of the form $A \otimes \cdots \otimes A$, for $A \in \mathrm{Sp}_{2n}\mathbb{C}$.

Proposition 17.20. *The algebra B is precisely the algebra of all endomorphisms of $V^{\langle d \rangle}$ commuting with all permutations in $\mathfrak{S}_d$.*

PROOF. If F is an endomorphism of $V^{\langle d \rangle}$ commuting with all permutations of factors, then the endomorphism $\tilde{F}$ of $V^{\otimes d}$ that is F on the factor $V^{\langle d \rangle}$ and zero on the complementary summand $\sum_I \Psi_I(V^{\otimes(d-2)})$ is an endomorphism that commutes with all permutations and all operators ϑ_I. The fact that $\widetilde{F}$ is a linear combination of operators from the symplectic group (which we know from Fact 17.19) implies the same for F. □

Corollary 17.21. *The representations* $\mathbb{S}_{\langle\lambda\rangle}(V)$ *are irreducible representations of* $\mathrm{Sp}_{2n}\mathbb{C}$.

PROOF. Since B is the commutator algebra to $A = \mathbb{C}[\mathfrak{S}_d]$ acting on the space $V^{\langle d\rangle}$, Lemma 6.22 implies that $(V^{\langle d\rangle})\cdot c_\lambda$ is an irreducible B-module. But we have seen that $(V^{\langle d\rangle})\cdot c_\lambda = \mathbb{S}_{\langle\lambda\rangle}V$, and the proposition shows that being irreducible over B is the same as being irreducible over $\mathrm{Sp}_{2n}\mathbb{C}$. □

Exercise 17.22*. Show that the multiplicity with which $\mathbb{S}_{\langle\lambda\rangle}(V)$ occurs in $V^{\langle d\rangle}$ is the dimension m_λ of the corresponding representation V_λ of $\mathfrak{S}_d$.

As was the case for the Weyl construction over $\mathrm{GL}_n\mathbb{C}$, there are general formulas for decomposing tensor products of these representations, as well as restrictions to subgroups $\mathrm{Sp}_{2n-2}\mathbb{C}$, and for their dimensions and multiplicities of weight spaces. We postpone these questions to Lecture 25, when we will have the Weyl character formula at our disposal.

As we saw in Lecture 15 for $\mathrm{GL}_n\mathbb{C}$, it is possible to make a commutative algebra which we denote by $\mathbb{S}^{\langle\bullet\rangle} = \mathbb{S}^{\langle\bullet\rangle}(V)$ out of the sum of all the irreducible representations of $\mathrm{Sp}_{2n}\mathbb{C}$, where $V = \mathbb{C}^{2n}$ is the standard representation. Probably the simplest way to do this, given what we have proved so far, is to start with the ring

$$\begin{aligned} A^\bullet(V, n) &= \mathrm{Sym}^\bullet(V \oplus \wedge^2 V \oplus \wedge^3 V \oplus \cdots \oplus \wedge^n V) \\ &= \bigoplus_{a_1,\ldots,a_n} \mathrm{Sym}^{a_n}(\wedge^n V) \otimes \cdots \otimes \mathrm{Sym}^{a_2}(\wedge^2 V) \otimes \mathrm{Sym}^{a_1}(V), \end{aligned}$$

the sum over all n-tuples $\mathbf{a} = (a_1, \ldots, a_n)$ of non-negative integers. Define a ring $\mathbb{S}^\bullet(V, n)$ to be the quotient of $A^\bullet(V, n)$ by the ideal generated by the same relations as in (15.53). By the argument in §15.5, the ring $\mathbb{S}^\bullet(V, n)$ is the direct sum of all the representations $\mathbb{S}_\lambda(V)$ of $\mathrm{GL}(V)$, as λ varies over all partitions with at most n parts.

The decomposition $V^{\otimes d} = V^{\langle d\rangle} \oplus W^{\langle d\rangle}$ of (17.12) determines a decomposition $V^{\otimes d}\cdot c_\lambda = V^{\langle d\rangle}\cdot c_\lambda \oplus W^{\langle d\rangle}\cdot c_\lambda$, which is a decomposition

$$\mathbb{S}_\lambda(V) = \mathbb{S}_{\langle\lambda\rangle}(V) \oplus J_{\langle\lambda\rangle}(V)$$

of representations of $\mathrm{Sp}_{2n}\mathbb{C}$. We claim that the sum $J^{\langle\bullet\rangle} = \bigoplus_\lambda J_{\langle\lambda\rangle}(V)$ is an ideal in $\mathbb{S}^\bullet(V, n) = \bigoplus_\lambda \mathbb{S}_\lambda(V)$. This is easy to see using weights, since $J_{\langle\lambda\rangle}(V)$ is the sum of all the representations in $\mathbb{S}_\lambda(V)$ whose highest weight is strictly smaller than λ. This implies that the image of $J_{\langle\lambda\rangle}(V) \otimes \mathbb{S}_\mu(V)$ in $\mathbb{S}_{\lambda+\mu}(V)$ is a sum of representations whose highest weights are less than $\lambda + \mu$, so they must be in $J_{\langle\lambda+\mu\rangle}(V)$.

The quotient ring is, therefore, the ring $\mathbb{S}^{\langle\bullet\rangle}(V)$ we were looking for:

$$\mathbb{S}^{\langle\bullet\rangle} = \mathbb{S}^{\bullet}(V, n)/J^{\langle\bullet\rangle} = \bigoplus_{\lambda} \mathbb{S}_{\langle\lambda\rangle}(V).$$

In fact, the ideal $J^{\langle\bullet\rangle}$ is generated by elements of the form $x \wedge \psi$, where $x \in \bigwedge^i V$, $i \leq n-2$, and ψ is the element in $\bigwedge^2 V$ corresponding to the skew form Q. An outline of the proof is sketched at the end of Lecture 25. The calculations, as well as other constructions of the ring, can be found in [L-T], where one can also find a discussion of functorial properties of the construction. For bases, see [DC-P], [L-M-S], and [M-S].

LECTURE 18

Orthogonal Lie Algebras

In this and the following two lectures we carry out for the orthogonal Lie algebras what we have already done in the special linear and symplectic cases. As in those cases, we start by working out in general the structure of the orthogonal Lie algebras, describing the roots, root spaces, Weyl group, etc., and then go to work on low-dimensional examples. There is one new phenomenon here: as it turns out, all three of the Lie algebras we deal with in §18.2 are isomorphic to symplectic or special linear Lie algebras we have already analyzed (this will be true of $\mathfrak{so}_6\mathbb{C}$ as well, but of no other orthogonal Lie algebra). As in the previous cases, the analysis of the Lie algebras and their representation theory will be completely elementary. Algebraic geometry does intrude into the discussion, however: we have described the isomorphisms between the orthogonal Lie algebras discussed and special linear and symplectic ones in terms of projective geometry, since that is what seems to us most natural. This should not be a problem; there are many other ways of describing these isomorphisms, and readers who disagree with our choice can substitute their own.

§18.1. $\mathrm{SO}_m\mathbb{C}$ and $\mathfrak{so}_m\mathbb{C}$

We will take up now the analysis of the Lie algebras of orthogonal groups. Here there is, as we will see very shortly, a very big difference in behavior between the so-called "even" orthogonal Lie algebras $\mathfrak{so}_{2n}\mathbb{C}$ and the "odd" orthogonal Lie algebras $\mathfrak{so}_{2n+1}\mathbb{C}$. Interestingly enough, the latter seem at first glance to be more complicated, especially in terms of notation; but when we analyze their representations we see that in fact they behave more regularly than the even ones. In any event, we will try to carry out the analysis in parallel fashion for as long as is feasible; when it becomes necessary to split up into cases, we will usually look at the even orthogonal Lie algebras first and then consider the odd.

Let V be a m-dimensional complex vector space, and

$$Q\colon V \times V \to \mathbb{C}$$

a nondegenerate, symmetric bilinear form on V. The orthogonal group $\mathrm{SO}_m\mathbb{C}$ is then defined to be the group of automorphisms A of V of determinant 1 preserving Q—that is, such that $Q(Av, Aw) = Q(v, w)$ for all $v, w \in V$—and the orthogonal Lie algebra $\mathfrak{so}_m\mathbb{C}$ correspondingly consists of endomorphisms $A\colon V \to V$ satisfying

$$Q(Av, w) + Q(v, Aw) = 0 \tag{18.1}$$

for all v and $w \in V$. As in the case of the symplectic Lie algebras, to carry out our analysis we want to write Q explicitly in terms of a basis for V, and here is where the cases of even and odd m first separate. In case $m = 2n$ is even, we will choose a basis for V in terms of which the quadratic form Q is given by

$$Q(e_i, e_{i+n}) = Q(e_{i+n}, e_i) = 1$$

and

$$Q(e_i, e_j) = 0 \quad \text{if } j \neq i \pm n.$$

The bilinear form Q may be expressed as

$$Q(x, y) = {}^t x \cdot M \cdot y,$$

where M is the $2n \times 2n$ matrix given in block form as

$$M = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix};$$

the group $\mathrm{SO}_{2n}\mathbb{C}$ is thus the group of $2n \times 2n$ matrices A with $\det(A) = 1$ and

$$M = {}^t A \cdot M \cdot A,$$

and the Lie algebra $\mathfrak{so}_{2n}\mathbb{C}$ correspondingly the space of matrices X satisfying the relation

$${}^t X \cdot M + M \cdot X = 0.$$

Writing a $2n \times 2n$ matrix X in block form as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

we have

$${}^t X \cdot M = \begin{pmatrix} {}^t C & {}^t A \\ {}^t D & {}^t B \end{pmatrix}$$

and

$$M \cdot X = \begin{pmatrix} C & D \\ A & B \end{pmatrix}$$

so that this relation is equivalent to saying that *the off-diagonal blocks B and C of X are skew-symmetric, and the diagonal blocks A and D of X are negative transposes of each other.*

Exercise 18.2. Show that with this choice of basis,

$$SO_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\} \cong \mathbb{C}^*,$$

and $\mathfrak{so}_2\mathbb{C} = \mathbb{C}$.

The situation in case the dimension m of V is odd is similar, if a little messier. To begin with, we will take Q to be expressible, in terms of a basis $e_1, \ldots, e_{2n+1}$ for V, by

$$Q(e_i, e_{i+n}) = Q(e_{i+n}, e_i) = 1 \quad \text{for } 1 \le i \le n;$$

$$Q(e_{2n+1}, e_{2n+1}) = 1;$$

and

$$Q(e_i, e_j) = 0 \quad \text{for all other pairs } i, j.$$

The bilinear form Q may be expressed as

$$Q(x, y) = {}^t x \cdot M \cdot y,$$

where M is the $(2n + 1) \times (2n + 1)$ matrix

$$M = \left(\begin{array}{c|c|c} 0 & I_n & 0 \\ \hline I_n & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

(the diagonal blocks here having widths n, n, and 1). The Lie algebra $\mathfrak{so}_{2n+1}\mathbb{C}$ is correspondingly the space of matrices X satisfying the relation ${}^tX \cdot M + M \cdot X = 0$; if we write X in block form as

$$X = \left(\begin{array}{c|c|c} A & B & E \\ \hline C & D & F \\ \hline G & H & J \end{array}\right),$$

then this is equivalent to saying that, as in the previous case, *B and C are skew-symmetric and A and D negative transposes of each other*; *and in addition $E = -{}^tH$, $F = -{}^tG$, and $J = 0$.*

With these choices, we may take as Cartan subalgebra—in both the even and odd cases—the subalgebra of matrices diagonal in this representation.[1]

[1] Note that if we had taken the simpler choice of Q, with M the identity matrix, the Lie algebra would have consisted of skew-symmetric matrices, and there would have been no nonzero diagonal matrices in the Lie algebra.

The subalgebra $\mathfrak{h}$ is thus generated by the n $2n \times 2n$ matrices $H_i = E_{i,i} - E_{n+i,n+i}$ whose action on V is to fix e_i, send e_{n+i} to its negative, and kill all the remaining basis vectors; note that this is the same whether $m = 2n$ or $2n + 1$. We will correspondingly take as basis for the dual vector space $\mathfrak{h}^*$ the dual basis L_j, where $\langle L_j, H_i \rangle = \delta_{i,j}$.

Given that the Cartan subalgebra of $\mathfrak{so}_{2n}\mathbb{C}$ coincides, as a subspace of $\mathfrak{sl}_{2n}\mathbb{C}$, with the Cartan subalgebra of $\mathfrak{sp}_{2n}\mathbb{C}$, we can use much of the description of the roots of $\mathfrak{sp}_{2n}\mathbb{C}$ to help locate the roots and root spaces of $\mathfrak{so}_{2n}\mathbb{C}$. For example, we saw in Lecture 16 that the endomorphism

$$X_{i,j} = E_{i,j} - E_{n+j,n+i} \in \mathfrak{sp}_{2n}\mathbb{C}$$

is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $L_i - L_j$. Since $X_{i,j}$ is also an element of $\mathfrak{so}_{2n}\mathbb{C}$, we see that $L_i - L_j$ is likewise a root of $\mathfrak{so}_{2n}\mathbb{C}$, with root space generated by $X_{i,j}$. Less directly but using the same analysis, we find that the endomorphisms

$$Y_{i,j} = E_{i,n+j} - E_{j,n+i}$$

and

$$Z_{i,j} = E_{n+i,j} - E_{n+j,i}$$

are eigenvectors for the action of $\mathfrak{h}$, with eigenvalues $L_i + L_j$ and $-L_i - L_j$, respectively (note that $Y_{i,j}$ and $Z_{i,j}$ do not coincide with their definitions in Lecture 16). In sum, then, *the roots of the Lie algebra $\mathfrak{so}_{2n}\mathbb{C}$ are the vectors* $\{\pm L_i \pm L_j\}_{i \neq j} \subset \mathfrak{h}^*$.

The case of the algebra $\mathfrak{so}_{2n+1}\mathbb{C}$ is similar; indeed, all the eigenvectors for the action of $\mathfrak{h}$ found above in $\mathfrak{so}_{2n}\mathbb{C}$, viewed as endomorphisms of $\mathbb{C}^{2n+1}$, are likewise eigenvectors for the action of $\mathfrak{h}$ on $\mathfrak{so}_{2n+1}\mathbb{C}$. In addition, we have the endomorphisms

$$U_i = E_{i,2n+1} - E_{2n+1,n+i}$$

and

$$V_i = E_{n+i,2n+1} - E_{2n+1,i}$$

which are eigenvectors with eigenvalues $+L_i$ and $-L_i$, respectively. The roots of $\mathfrak{so}_{2n+1}\mathbb{C}$ are thus the roots $\pm L_i \pm L_j$ of $\mathfrak{so}_{2n}\mathbb{C}$, together with additional roots $\pm L_i$.

We note that we could have arrived at these statements without decomposing the Lie algebras $\mathfrak{so}_m\mathbb{C}$: the description (18.1) of the orthogonal Lie algebra may be interpreted as saying that, in terms of the identification of V with V^* given by the form Q, $\mathfrak{so}_m\mathbb{C}$ is just the Lie algebra of skew-symmetric endomorphisms of V (an endomorphism being skew-symmetric if it is equal to minus its transpose). That is, the adjoint representation of $\mathfrak{so}_m\mathbb{C}$ is isomorphic to the wedge product $\wedge^2 V$. In the even case $m = 2n$, since the weights of V are $\pm L_i$ (inasmuch as the subalgebras $\mathfrak{h} \subset \mathrm{End}(V)$ coincide, the weights of V must likewise be the same for $\mathfrak{so}_{2n}\mathbb{C}$ as for $\mathfrak{sp}_{2n}\mathbb{C}$), it follows that the roots of $\mathfrak{so}_{2n}\mathbb{C}$

are just the pairwise distinct sums $\pm L_i \pm L_j$. In the odd case $m = 2n + 1$, we see that $e_{2n+1} \in V$ is an eigenvector for the action of $\mathfrak{h}$ with eigenvalue 0, so that the weights of the standard representation V are $\{\pm L_i\} \cup \{0\}$ and the weights of the adjoint representation correspondingly $\{\pm L_i \pm L_j\} \cup \{\pm L_i\}$.

Exercise 18.3. Use a similar analysis to find the roots of $\mathfrak{sp}_{2n}\mathbb{C}$ without explicit calculation.

To make a comparison with the Lie algebra $\mathfrak{sp}_{2n}\mathbb{C}$, we can say that the root diagram of $\mathfrak{so}_{2n}\mathbb{C}$ looks like that of $\mathfrak{sp}_{2n}\mathbb{C}$ with the roots $\pm 2L_i$ removed, whereas the root diagram of $\mathfrak{so}_{2n+1}\mathbb{C}$ looks like that of $\mathfrak{sp}_{2n}\mathbb{C}$ with the roots $\pm 2L_i$ replaced by $\pm L_i$. Note that this immediately tells us what the Weyl groups are: first, in the case of $\mathfrak{so}_{2n+1}\mathbb{C}$, the Weyl group is the same as that of $\mathfrak{sp}_{2n}\mathbb{C}$:

$$1 \to (\mathbb{Z}/2)^n \to \mathfrak{W}_{\mathfrak{so}_{2n+1}\mathbb{C}} \to \mathfrak{S}_n \to 1.$$

In the case of $\mathfrak{so}_{2n}\mathbb{C}$, the Weyl group is the subgroup of the Weyl group of $\mathfrak{sp}_{2n}\mathbb{C}$ generated by reflection in the hyperplanes perpendicular to the roots $\pm L_i \pm L_j$, without the additional generator given by reflection in the roots $\pm L_i$. This subgroup still acts as the full symmetric group on the set of coordinate axes in $\mathfrak{h}^*$; but the kernel of this action, instead of acting as $\pm I$ on each of the coordinate axes independently, will consist of transformations of determinant 1; i.e., will act as -1 on an even number of axes. (That every such transformation is indeed in the Weyl group is easy to see: for example, reflection in the plane perpendicular to $L_i + L_j$ followed by reflection in the plane perpendicular to $L_i - L_j$ will send L_i to $-L_i$, L_j to $-L_j$, and L_k to L_k for $k \neq i, j$.) Another way to say this is that the Weyl group is the subgroup of the Weyl group of $\mathfrak{sp}_{2n}\mathbb{C}$ consisting of transformations whose determinant agrees with the sign of the induced permutation of the coordinate axes; so that while the Weyl group of $\mathfrak{sp}_{2n}\mathbb{C}$ fits into the exact sequence

$$1 \to (\mathbb{Z}/2)^n \to \mathfrak{W}_{\mathfrak{sp}_{2n}\mathbb{C}} \to \mathfrak{S}_n \to 1,$$

the Weyl group of $\mathfrak{so}_{2n}\mathbb{C}$ has instead the sequence

$$1 \to (\mathbb{Z}/2)^{n-1} \to \mathfrak{W}_{\mathfrak{so}_{2n}\mathbb{C}} \to \mathfrak{S}_n \to 1.$$

We can likewise describe the Weyl chambers of $\mathfrak{so}_{2n}\mathbb{C}$ and $\mathfrak{so}_{2n+1}\mathbb{C}$ by direct comparison with $\mathfrak{sp}_{2n}\mathbb{C}$. To start, to choose an ordering of the roots we take as linear functional on $\mathfrak{h}^*$ a form $l = c_1H_1 + \cdots + c_nH_n$, where $c_1 > c_2 > \cdots > c_n > 0$. The positive roots in the case of $\mathfrak{so}_{2n+1}\mathbb{C}$ are then

$$R^+ = \{L_i + L_j\}_{i<j} \cup \{L_i - L_j\}_{i<j} \cup \{L_i\}_i,$$

whereas in the case of $\mathfrak{so}_{2n}\mathbb{C}$ we have

$$R^+ = \{L_i + L_j\}_{i<j} \cup \{L_i - L_j\}_{i<j}.$$

The primitive positive roots are

$$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, L_n \qquad \text{for } \mathfrak{so}_{2n+1}\mathbb{C};$$
$$L_1 - L_2, L_2 - L_3, \ldots, L_{n-1} - L_n, L_{n-1} + L_n \quad \text{for } \mathfrak{so}_{2n}\mathbb{C}.$$

In the first case, the Weyl chamber is exactly the same as for $\mathfrak{sp}_{2n}\mathbb{C}$, namely, for $m = 2n + 1$,

$$\mathscr{W} = \{\sum a_i L_i \colon a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\}$$

since the roots are the same except for the factor of 2 on some. In the case of $\mathfrak{so}_{2n}\mathbb{C}$, since there is no root along the line spanned by the L_i, the equality $a_n = 0$ does not describe a face of the Weyl chamber; however, since $L_{n-1} + L_n$ is still a root (and a positive one) we still have the inequality $a_{n-1} + a_n \geq 0$ in $\mathscr{W}$, so that we can write, for $m = 2n$,

$$\mathscr{W} = \{\sum a_i L_i \colon a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq |a_n|\}.$$

(Note that in the case of $\mathfrak{so}_{2n}\mathbb{C}$ we could have chosen our linear functional $l = c_1 H_1 + \cdots + c_n H_n$ with $c_1 > c_2 > \cdots > -c_n > 0$; the ordering of the roots, and consequently the Weyl chamber, would still be the same.)

As for the Killing form, the same considerations as for the symplectic case show that it must be, up to scalars, the standard quadratic form: $B(H_i, H_j) = \delta_{i,j}$. (This was implicit in the above description of the Weyl group.) The explicit calculation is no more difficult, and we leave it as an exercise:

$$B(\sum a_i H_i, \sum b_i H_i) = \begin{cases} (4n-2)\sum a_i b_i & \text{if } m = 2n+1 \\ (4n-4)\sum a_i b_i & \text{if } m = 2n. \end{cases}$$

Next, to describe the representations of the orthogonal Lie algebras we have to determine the weight lattice in $\mathfrak{h}^*$; and to do this we must, as before, locate the copies $\mathfrak{s}_\alpha$ of $\mathfrak{sl}_2\mathbb{C}$ corresponding to the root pairs $\pm\alpha$, and the corresponding distinguished elements H_α of $\mathfrak{h}$. This is so similar to the case of $\mathfrak{sp}_{2n}\mathbb{C}$ that we will leave the actual calculations as an exercise; we will simply state here the results that in $\mathfrak{so}_m\mathbb{C}$ for any m,

(i) the distinguished copy $\mathfrak{s}_{L_i - L_j}$ of $\mathfrak{sl}_2\mathbb{C}$ associated to the root $L_i - L_j$ is the span of the root spaces $\mathfrak{g}_{L_i - L_j} = \mathbb{C}\cdot X_{i,j}$, $\mathfrak{g}_{-L_i - L_j} = \mathbb{C}\cdot X_{j,i}$ and their commutator $[X_{i,j}, X_{j,i}] = E_{i,i} - E_{j,j} + E_{n+j,n+j} - E_{n+i,n+i}$, with distinguished element $H_{L_i - L_j} = H_i - H_j$ (this is exactly as in the case of $\mathfrak{sp}_{2n}\mathbb{C}$);

(ii) the distinguished copy $\mathfrak{s}_{L_i + L_j}$ of $\mathfrak{sl}_2\mathbb{C}$ associated to the root $L_i + L_j$ is the span of the root spaces $\mathfrak{g}_{L_i + L_j} = \mathbb{C}\cdot Y_{i,j}$, $\mathfrak{g}_{-L_i - L_j} = \mathbb{C}\cdot Z_{i,j}$ and their commutator $[Y_{i,j}, Z_{i,j}] = -E_{i,i} + E_{j,j} - E_{n+j,n+j} + E_{n+i,n+i} = -H_i - H_j$, with distinguished element $H_{L_i + L_j} = H_i + H_j$ (so that we have also $H_{-L_i - L_j} = -H_i - H_j$); and in the case of $\mathfrak{so}_{2n+1}\mathbb{C}$,

(iii) the distinguished copy $\mathfrak{s}_{L_i}$ of $\mathfrak{sl}_2\mathbb{C}$ associated to the root L_i is the span of the root spaces $\mathfrak{g}_{L_i} = \mathbb{C}\cdot U_i$, $\mathfrak{g}_{-L_i} = \mathbb{C}\cdot V_i$ and their commutator $[U_i, V_i] = [E_{i,2n+1} - E_{2n+1,n+i}, E_{n+i,2n+1} - E_{2n+1,i}] = -H_i$, with distinguished element $H_{L_i} = 2H_i$ (so that $H_{-L_i} = -2H_i$ as well).

Exercise 18.4. Verify the computations made here.

Again, the configuration of distinguished elements resembles that of $\mathfrak{sp}_{2n}\mathbb{C}$ closely; that of $\mathfrak{so}_{2n+1}\mathbb{C}$ differs from it by the substitution of $\pm 2H_i$ for $\pm H_i$, whereas that of $\mathfrak{so}_{2n}\mathbb{C}$ differs by the removal of the $\pm H_i$. The effect on the weight lattice is the same in either case: *for both even and odd orthogonal Lie algebras, the weight lattice Λ_W is the lattice generated by the L_i together with the element $(L_1 + \cdots + L_n)/2$.*

Exercise 18.5. Show that

$$\Lambda_W/\Lambda_R = \begin{cases} \mathbb{Z}/2 & \text{if } m = 2n+1 \\ \mathbb{Z}/4 & \text{if } m = 2n \text{ and } n \text{ is odd} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } m = 2n \text{ and } n \text{ is even.} \end{cases}$$

§18.2. Representations of $\mathfrak{so}_3\mathbb{C}$, $\mathfrak{so}_4\mathbb{C}$, and $\mathfrak{so}_5\mathbb{C}$

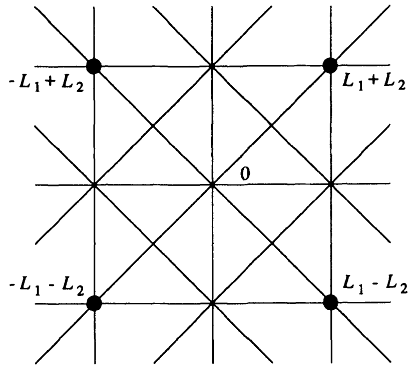
To give some examples, start with the case $n = 1$. Of course, $\mathfrak{so}_2\mathbb{C} \cong \mathbb{C}$ is not semisimple. The root system of $\mathfrak{so}_3\mathbb{C}$, on the other hand, looks like that of $\mathfrak{sl}_2\mathbb{C}$:

This is because, in fact, the two Lie algebras are isomorphic. Indeed, like the symplectic group, the quotient $PSO_m\mathbb{C}$ of the orthogonal group by its center can be realized as the motions of the projective space $\mathbb{P}V$ preserving isotropic subspaces for the quadratic form Q; in particular, this means we can realize $PSO_m\mathbb{C}$ as the group of motions of $\mathbb{P}V = \mathbb{P}^{m-1}$ carrying the quadric hypersurface

$$\bar{Q} = \{[v]: Q(v, v) = 0\}$$

into itself. In the first case of this, we see that the group $PSO_3\mathbb{C}$ is the group of motions of the projective plane $\mathbb{P}^2$ carrying a conic curve $C \subset \mathbb{P}^2$ into itself. But we have seen before that this group is also $PGL_2\mathbb{C}$ (the conic curve is itself isomorphic to $\mathbb{P}^1$, and the group acts as its full group of automorphisms), giving us the isomorphism $\mathfrak{so}_3\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$. One thing to note here is that the "standard" representation of $\mathfrak{so}_3\mathbb{C}$ is not the standard representation of $\mathfrak{sl}_2\mathbb{C}$, but rather its symmetric square. In fact, the irreducible representation with highest weight $\frac{1}{2}L_1$ is not contained in tensor powers of the standard representation of $\mathfrak{so}_3\mathbb{C}$. This will turn out to be significant: the standard representation of $\mathfrak{sl}_2\mathbb{C}$, viewed as a representation of $\mathfrak{so}_3\mathbb{C}$, is the first example of a *spin* representation of an orthogonal Lie algebra.

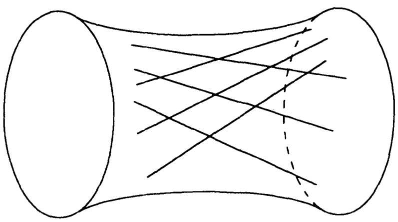
The next examples involve two-dimensional Cartan algebras. First we have $\mathfrak{so}_4\mathbb{C}$, whose root diagram looks like



Note one thing about this diagram: the roots are located on the union of two complementary lines. This says, by Exercise 14.33, that the Lie algebra $\mathfrak{so}_4\mathbb{C}$ is decomposable, and in fact should be the sum of two algebras each of whose root diagrams looks like that of $\mathfrak{sl}_2\mathbb{C}$; explicitly, $\mathfrak{so}_4\mathbb{C}$ is the direct sum of the two algebras $\mathfrak{s}_\alpha$, for $\alpha = L_1 + L_2$ and $\alpha = L_1 - L_2$. In fact, we can see this isomorphism

$$\mathfrak{so}_4\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}, \tag{18.6}$$

as in the previous example, geometrically. Precisely, we may realize the group $\mathrm{PSO}_4\mathbb{C} = \mathrm{SO}_4\mathbb{C}/\{\pm I\}$ as the connected component of the identity in the group of motions of projective three-space $\mathbb{P}^3$ carrying a quadric hypersurface $\bar{Q}$ into itself. But a quadric hypersurface in $\mathbb{P}^3$ has two rulings by lines, and these two rulings give an isomorphism of $\bar{Q}$ with a product $\mathbb{P}^1 \times \mathbb{P}^1$

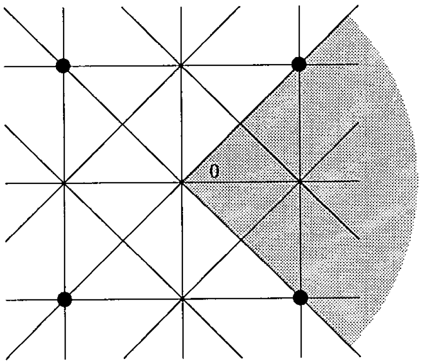


$\mathrm{PSO}_4\mathbb{C}$ thus acts on the product $\mathbb{P}^1 \times \mathbb{P}^1$; and since the connected component of the identity in the automorphism group of this variety is just the product $\mathrm{PGL}_2\mathbb{C} \times \mathrm{PGL}_2\mathbb{C}$, we get an inclusion

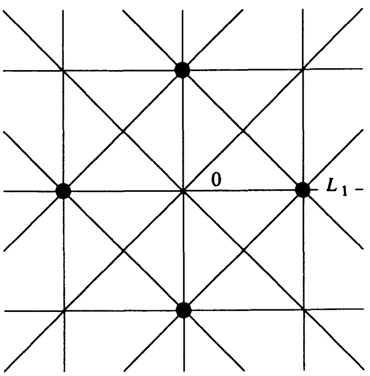
$$\mathrm{PSO}_4\mathbb{C} \to \mathrm{PGL}_2\mathbb{C} \times \mathrm{PGL}_2\mathbb{C}.$$

Another way of saying this is to remark that $\mathrm{PSO}_4\mathbb{C}$ acts on the variety of isotropic 2-planes for the quadratic form Q on V; and this variety is just the disjoint union of two copies of $\mathbb{P}^1$. To see in this case that the map is an isomorphism, consider the tensor product $V = U \otimes W$ of the pullbacks to $\mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$ of the standard representations of the two factors. Clearly the action on $\mathbb{P}(U \otimes W)$ will preserve the points corresponding to decomposable tensors (that is, points of the form $[u \otimes w]$); but the locus of such points is just a quadric hypersurface, giving us the inverse inclusion of $\mathrm{PGL}_2\mathbb{C} \times \mathrm{PGL}_2\mathbb{C}$ in $\mathrm{PSO}_4\mathbb{C}$.

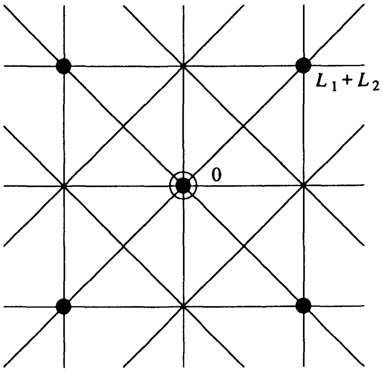
In fact, all of this will fall out of the analysis of the representations of $\mathfrak{so}_4\mathbb{C}$, if we just pursue it as usual. To begin with, the Weyl chamber we have selected looks like



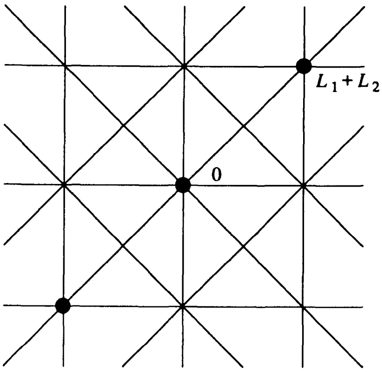
Now, the standard representation has, as noted above, weight diagram



with highest weight L_1 (note that the highest weight of the standard representation lies in this case in the interior of the Weyl chamber, something of an anomaly). Its second exterior power will have weights $\pm L_1 \pm L_2$ and 0 (occurring with multiplicity 2), i.e., diagram

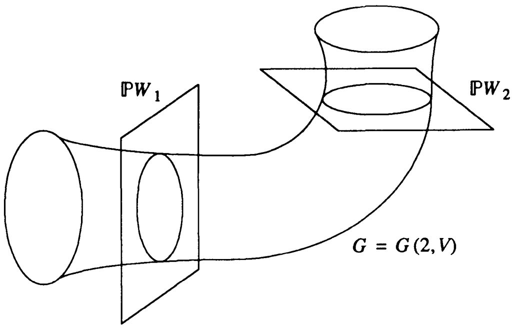


We see one thing about this representation right away, namely, that it cannot be irreducible. Indeed, the images of the highest weight $L_1 + L_2$ under the Weyl group consist just of $\pm(L_1 + L_2)$, so that the diagram of the irreducible representation with this highest weight is



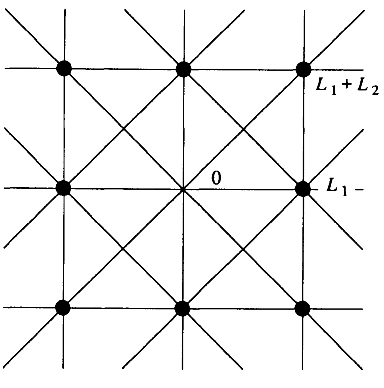
We see from this that the second exterior power $\wedge^2 V$ of the standard representation of $\mathfrak{so}_4\mathbb{C}$ must be the direct sum of the irreducible representations $W_1 = \Gamma_{L_1+L_2}$ and $W_2 = \Gamma_{L_1-L_2}$ with highest weights $L_1 + L_2$ and $L_1 - L_2$. Since $\wedge^2 V$ is at the same time the adjoint representation, this says that $\mathfrak{so}_4\mathbb{C}$ itself must be a product of Lie algebras with adjoint representations $\Gamma_{L_1+L_2}$ and $\Gamma_{L_1-L_2}$.

One way to derive the picture of the ruling of the quadric in $\mathbb{P}^3$ from this decomposition is to view $\mathfrak{so}_4\mathbb{C}$ as a subalgebra of $\mathfrak{sl}_4\mathbb{C}$, and the action of $PSO_4\mathbb{C}$ on $\mathbb{P}(\wedge^2 V)$ as a subgroup of the group of motions of $\mathbb{P}^2(\wedge^2 V) = \mathbb{P}^5$ preserving the Grassmannian $G = G(2, V)$ of lines in $\mathbb{P}^3$. In fact, we see from the above that the action of PSO_4 on $\mathbb{P}^5$ will preserve a pair of complementary 2-planes $\mathbb{P}W_1$ and $\mathbb{P}W_2$; it follows that this action must carry into themselves the intersections of these 2-planes with the Grassmannian. These intersections are conic curves, corresponding to one-parameter families of lines sweeping out a quadric surface (necessarily the same quadric, since the action of $SO_4\mathbb{C}$ on V preserves a unique quadratic form); thus, the two rulings of the quadric.



Note one more aspect of this example: as in the case of $\mathfrak{so}_3\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C}$, the weights of the standard representation of $\mathfrak{so}_4\mathbb{C}$ do not generate the weight lattice, but rather a sublattice $\mathbb{Z}\{L_1, L_2\}$ of index 2 in Λ_W. Thus, there is no way of constructing all the representations of $\mathfrak{so}_4\mathbb{C}$ by applying linear- or multilinear-algebraic constructions to the standard representation; it is only after we are aware of the isomorphism $\mathfrak{so}_4\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$ that we can construct, for example, the representation $\Gamma_{(L_1+L_2)/2}$ with highest weight $(L_1 + L_2)/2$ (of course, this is just the pullback from the first factor of $\mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$ of the standard representation of $\mathfrak{sl}_2\mathbb{C}$).

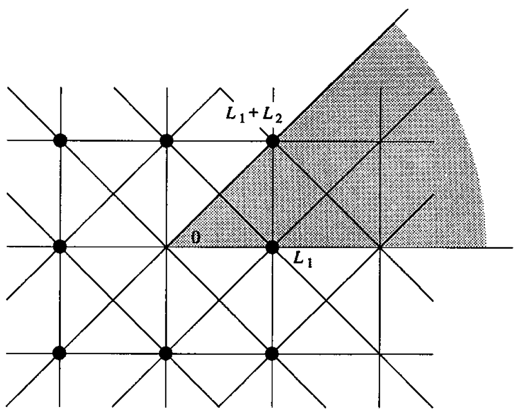
We come now to the case of $\mathfrak{so}_5\mathbb{C}$, which is more interesting. The root diagram in this case looks like



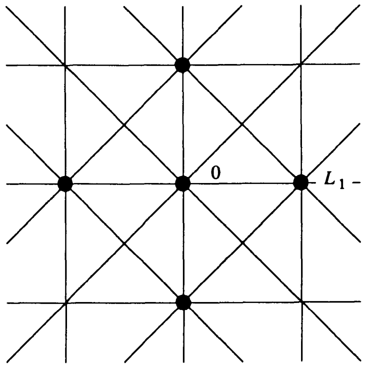
(as in the preceding example, the weight lattice is the lattice of intersections of all the lines drawn). The first thing we should notice about this diagram is that it is isomorphic to the weight diagram of the Lie algebra $\mathfrak{sp}_4\mathbb{C}$; the diagram just appears here rotated through an angle of $\pi/4$. Indeed, this is not accidental; the two Lie algebras $\mathfrak{sp}_4\mathbb{C}$ and $\mathfrak{so}_5\mathbb{C}$ are isomorphic, and it is not hard to construct this isomorphism explicitly. To see the isomorphism geometrically, we simply have to recall the identification, made in Lecture 14, of the group $\mathrm{PSp}_4\mathbb{C}$ with a group of motions of $\mathbb{P}^4$. There, we saw that the larger group $\mathrm{PGL}_4\mathbb{C}$ could be identified with the automorphisms of the projective space $\mathbb{P}(\wedge^2 V) = \mathbb{P}^5$ preserving the Grassmannian $G = G(2, 4) \subset \mathbb{P}(\wedge^2 V)$. The subgroup $\mathrm{PSp}_4\mathbb{C} \subset \mathrm{PGL}_4\mathbb{C}$ thus preserves both the Grassmannian G, which is a quadric hypersurface in $\mathbb{P}^5$, and the decomposition of $\wedge^2 V$ into the span $\mathbb{C}\cdot Q$ of the skew form $Q \in \wedge^2 V^* \cong \wedge^2 V$ and its complement W, and so acts on $\mathbb{P}W$ carrying the intersection $G_L = G \cap \mathbb{P}W$ into itself. We thus saw that $\mathrm{PSp}_4\mathbb{C}$ was a subgroup of the group of motions of projective space $\mathbb{P}^4$ preserving a quadric hypersurface, and asserted that in fact it was the whole group.

(To see the reverse inclusion directly, we can invoke a little algebraic geometry, which tells us that the locus of isotropic lines for a quadric in $\mathbb{P}^4$ is isomorphic to $\mathbb{P}^3$, so that $\mathrm{PSO}_5\mathbb{C}$ acts on $\mathbb{P}^3$. Moreover, this action preserves the subset of pairs of points in $\mathbb{P}^3$ whose corresponding lines in $\mathbb{P}^4$ intersect, which, for a suitably defined skew-symmetric bilinear form $\tilde{Q}$, is exactly the set of pairs $([v], [w])$ such that $\tilde{Q}(v, w) = 0$, so that we have an inclusion of $\mathrm{PSO}_5\mathbb{C}$ in $\mathrm{PSp}_4\mathbb{C}$.)

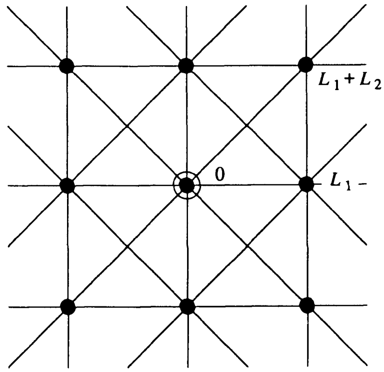
Let us proceed to analyze the representations of $\mathfrak{so}_5\mathbb{C}$ as we would ordinarily, bearing in mind the isomorphism with $\mathfrak{sp}_4\mathbb{C}$. To begin with, we draw the Weyl chamber picked out above in $\mathfrak{h}^*$:



As for the representations of $\mathfrak{so}_5\mathbb{C}$, we have to begin with the standard, which has weight diagram



This we see corresponds to the representation $W = \bigwedge^2 V/\mathbb{C}\cdot Q$ of $\mathfrak{sp}_4\mathbb{C}$. Next, the second exterior power of the standard representation of $\mathfrak{so}_5\mathbb{C}$ has weights



This is of course the adjoint representation of $\mathfrak{so}_5\mathbb{C}$; it is the irreducible representation with highest weight $L_1 + L_2$. Note that it corresponds to the symmetric square $\mathrm{Sym}^2 V$ of the standard representation of $\mathfrak{sp}_4\mathbb{C}$ (see Exercise 16.8).

Exercise 18.7. Show that contraction with the quadratic form $Q \in \mathrm{Sym}^2 V^*$ preserved by the action of $\mathfrak{so}_5\mathbb{C}$ induces maps

$$\varphi\colon \mathrm{Sym}^a V \to \mathrm{Sym}^{a-2} V.$$

Show that the kernel of this contraction is exactly the irreducible representation with highest weight $a\cdot L_1$. Compare this with the analysis in Exercise 16.11.

Exercise 18.8. Examine the symmetric power $\mathrm{Sym}^a(\wedge^2 V)$ of the representation $\wedge^2 V$. This will contain a copy of the irreducible representation $\Gamma_{a(L_1+L_2)}$; what else will it contain? Interpret these other factors in light of the isomorphism $\mathfrak{so}_5\mathbb{C} \cong \mathfrak{sp}_4\mathbb{C}$.

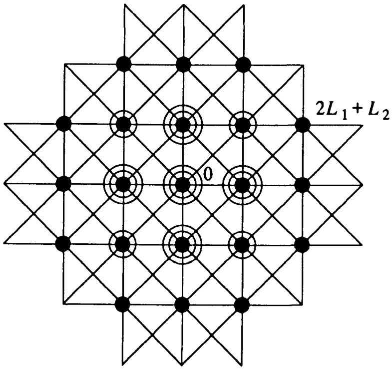
Exercise 18.9. For an example of a "mixed" tensor, consider the irreducible representation $\Gamma_{2L_1+L_2}$. Show that this is contained in the kernels of the wedge product map

$$\varphi: V \otimes \wedge^2 V \to \wedge^3 V$$

and the composition

$$\varphi': V \otimes \wedge^2 V \to V^* \otimes \wedge^2 V \to V,$$

where the first map is induced by the isomorphism $\tilde{Q}: V \to V^*$ and the second is the contraction $V^* \otimes \wedge^2 V \to V$. Is it equal to the intersection of these kernels? Show that the weight diagram of this representation is



After you are done with this analysis, compare with the analysis given of the corresponding representation in Lecture 16.

Note that, as in the case of the other orthogonal Lie algebras studied so far (and as is the case for all $\mathfrak{so}_m\mathbb{C}$), *the weights of the standard representation do not generate the weight lattice, but only the sublattice of index two generated by the L_i*. Thus, the tensor algebra of the standard representation will contain only one-half of all the irreducible representations of $\mathfrak{so}_5\mathbb{C}$. Now, we do know that there are others, and even something about them—for example, we see in the following exercise that the irreducible representation of $\mathfrak{so}_5\mathbb{C}$ with highest weight $(L_1 + L_2)/2$ is a sort of "symmetric square root" of the adjoint representation:

Exercise 18.10. Show, using only root and weight diagrams for $\mathfrak{so}_5\mathbb{C}$, that the exterior square $\wedge^2 V$ of the standard representation of $\mathfrak{so}_5\mathbb{C}$ is actually the symmetric square of an irreducible representation.

We can also describe this irreducible representation via the isomorphism of $\mathfrak{so}_5\mathbb{C}$ with $\mathfrak{sp}_4\mathbb{C}$: it is just the standard representation of $\mathfrak{sp}_4\mathbb{C}$ on $\mathbb{C}^4$. We do not at this point have, however, a way of constructing this representation without invoking the isomorphism. This representation, the representation of $\mathfrak{so}_3\mathbb{C}$ with highest weight $L_1/2$, and the representation of $\mathfrak{so}_4\mathbb{C}$ with highest weight $(L_1 + L_2)/2$ discussed above are called *spin* representations of the corresponding Lie algebras and will be the subject matter of Lecture 20.

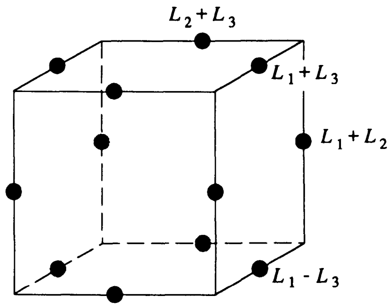
LECTURE 19

$\mathfrak{so}_6\mathbb{C}$, $\mathfrak{so}_7\mathbb{C}$, and $\mathfrak{so}_m\mathbb{C}$

This lecture is analogous in content (and prerequisites) to Lecture 17: we do some more low-dimensional examples and then describe the general picture of the representations of the orthogonal Lie algebras. One difference is that only half the irreducible representations of $\mathfrak{so}_m\mathbb{C}$ lie in the tensor algebra of the standard; to complete the picture of the representation theory we have to construct the spin representations, which is the subject matter of the following lecture. The first four sections are completely elementary (except possibly for the discussion of the isomorphism $\mathfrak{so}_6\mathbb{C} \cong \mathfrak{sl}_4\mathbb{C}$ in §19.1); the last section assumes a knowledge of Lecture 6 and §15.3, but can be skipped by those who did not read those sections.

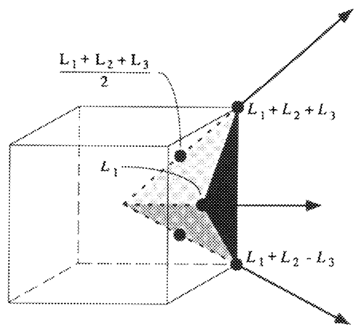
§19.1. Representations of $\mathfrak{so}_6\mathbb{C}$

We continue our discussion of orthogonal Lie algebras with the example of $\mathfrak{so}_6\mathbb{C}$. First, its root diagram:

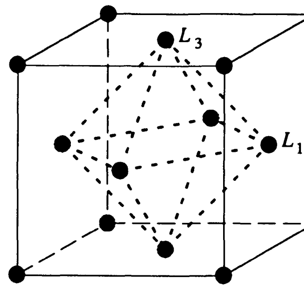


Once more (and for the last time), we notice a coincidence between this and the root diagram of a Lie algebra already studied, namely, $\mathfrak{sl}_4\mathbb{C}$. In fact, the two Lie algebras are isomorphic. The isomorphism is one we have already observed, in a sense: in the preceding lecture we noted that if V is a four-dimensional vector space, then the group $PGL_4\mathbb{C}$ may be realized as the connected component of the identity in the group of motions of $\mathbb{P}(\wedge^2 V) = \mathbb{P}^5$ carrying the Grassmannian $G = G(2, 4) \subset \mathbb{P}(\wedge^2 V)$ into itself, and $PSp_4\mathbb{C} \subset PGL_4\mathbb{C}$ the subgroup fixing a hyperplane $\mathbb{P}W = \mathbb{P}^4 \subset \mathbb{P}^5$. We used this to identify the subgroup $PSp_4\mathbb{C}$ with the orthogonal group $PSO_5\mathbb{C}$; at the same time it gives an identification of the larger group $PGL_4\mathbb{C}$ with the orthogonal group $PSO_6\mathbb{C}$.

Even though $\mathfrak{so}_6\mathbb{C}$ is isomorphic to a Lie algebra we have already examined, it is worth going through the analysis of its representations for what amounts to a second time, partly so as to understand the isomorphism better, but mainly because we will see clearly in the case of $\mathfrak{so}_6\mathbb{C}$ a number of phenomena that will hold true of the even orthogonal groups in general. To start, we draw the Weyl chamber in $\mathfrak{h}^*$:

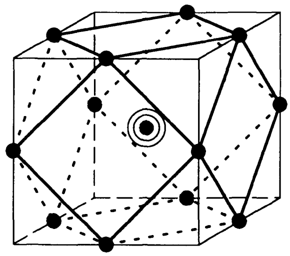


As usual, we begin with the standard representation, which has weights $\pm L_i$, corresponding to the centers of the faces of the cube:



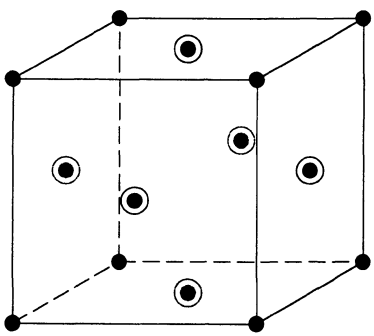
Note that the highest weight L_1 once more lies on an edge of the Weyl chamber (the front edge, in the diagram on the preceding page). Observe that the standard representation of $\mathfrak{so}_6\mathbb{C}$ corresponds, as we have already pointed out, to the exterior square of the standard representation of $\mathfrak{sl}_4\mathbb{C}$.

Next, we look at the exterior square $\bigwedge^2 V$ of the standard representation of $\mathfrak{so}_6\mathbb{C}$. This will have weights $\pm L_i \pm L_j$ (of course, it is the adjoint representation) and so will have weight diagram

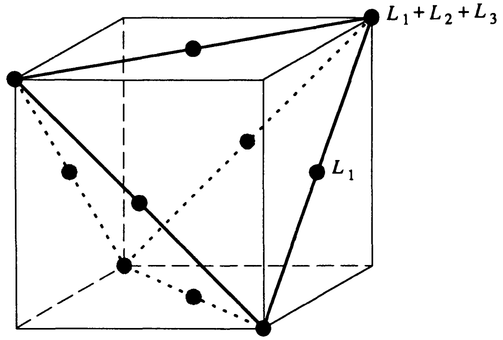


Note that the highest weight vector $L_1 + L_2$ of this representation does not lie on an edge of the Weyl chamber, but rather in the interior of a face (the back face, in the diagram above). In order to generate all the representations, we still need to find the irreducible representations with highest weight along the remaining two edges of the Weyl chamber.

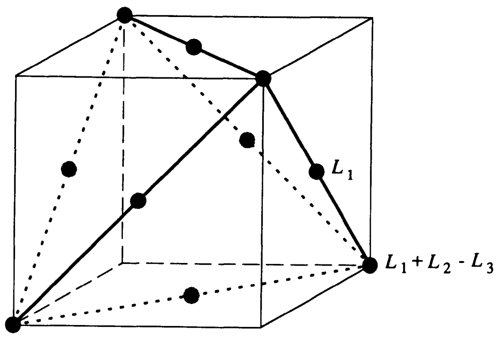
We look next at the exterior cube $\bigwedge^3 V$ of the standard representation. The weights here are the eight weights $\pm L_1 \pm L_2 \pm L_3$, each taken with multiplicity one, and the six weights $\pm L_i$, each taken with multiplicity 2, as in the diagram



Now, we notice something very interesting: this cannot be an irreducible representation. We can see this in a number of ways: the images of the weight $L_1 + L_2 + L_3$ under the Weyl group, for example, consist of every other vertex of the reference cube; in particular, their convex hull does not contain the remaining four vertices including $L_1 + L_2 - L_3$. Equivalently, there is no way to go from $L_1 + L_2 + L_3$ to $L_1 + L_2 - L_3$ by translation by negative root vectors. The representation $\bigwedge^3 V$ will thus contain copies of the irreducible representations $\Gamma_{L_1+L_2+L_3}$ and $\Gamma_{L_1+L_2-L_3}$ with highest weights $L_1 + L_2 + L_3$ and $L_1 + L_2 - L_3$, with weight diagrams



and



Since the weight diagram of each of these is a tetrahedron containing the weights $\pm L_i$, we have accounted for all the weights of $\bigwedge^3 V$ and so must have a direct sum decomposition

$$\bigwedge^3 V = \Gamma_{L_1+L_2+L_3} \oplus \Gamma_{L_1+L_2-L_3}.$$

We can relate this direct sum decomposition to a geometric feature of a quadric hypersurface in $\mathbb{P}^5$, analogous to the presence of two rulings on a quadric in $\mathbb{P}^3$. We saw before that the locus of lines lying on a quadric surface in $\mathbb{P}^3$ turns out to be disconnected, consisting of two components each isomorphic to $\mathbb{P}^1$ (and embedded, via the Plücker embedding of the Grassmannian $G = G(2, 4)$ of lines in $\mathbb{P}^3$ in $\mathbb{P}(\wedge^2\mathbb{C}) = \mathbb{P}^5$, as a pair of conic curves lying in complementary 2-planes in $\mathbb{P}^5$). In a similar fashion, the variety of 2-planes lying on a quadric hypersurface in $\mathbb{P}^5$ turns out to be disconnected, consisting of two components that, under the Plücker embedding of $G(3, 6)$ in $\mathbb{P}(\wedge^3\mathbb{C}^6) = \mathbb{P}^{19}$, span two complementary 9-planes $\mathbb{P}W_1$ and $\mathbb{P}W_2$; these two planes give the direct sum decomposition of $\wedge^3 V$ as an $\mathfrak{so}_6\mathbb{C}$-module.

In fact, if we think of a quadric hypersurface in $\mathbb{P}^5$ as the Grassmannian $G = G(2, 4)$ of lines in $\mathbb{P}^3$, we can see explicitly what these two families of 2-planes are: for every point $p \in \mathbb{P}^3$ the locus of lines passing through p forms a 2-plane on G, and for every plane $H \subset \mathbb{P}^3$ the locus of lines lying in H is a 2-plane in G. These are the two families; indeed, in this case we can go two steps further. First, we see from this that each of these families is parametrized by $\mathbb{P}^3$, so that the connected component $\mathrm{PSO}_6\mathbb{C}$ of the identity in the group of motions of $\mathbb{P}^5$ preserving the Grassmannian acts on $\mathbb{P}^3$, giving us the inverse inclusion $\mathrm{PSO}_6\mathbb{C} \subset \mathrm{PGL}_4\mathbb{C}$. Second, under the Plücker embedding each of these families is carried into a copy of the quadratic Veronese embedding of $\mathbb{P}^3$ into $\mathbb{P}^9$, giving us the identification of the direct sum factors of the third exterior power of the standard representation of $\mathfrak{so}_6\mathbb{C}$ with the symmetric square of the standard representation of $\mathfrak{sl}_4\mathbb{C}$.

Exercise 19.1. Verify, without using the isomorphism with $\mathfrak{so}_6\mathbb{C}$ and the analysis above, that the standard representation V of $\mathfrak{sl}_4\mathbb{C}$ satisfies

$$\wedge^3(\wedge^2 V) \cong \mathrm{Sym}^2 V \oplus \mathrm{Sym}^2 V^*.$$

Note that we have now identified, in terms of tensor powers of the standard one, irreducible representations of $\mathfrak{so}_6\mathbb{C}$ with highest weight vectors L_1, $L_1 + L_2 + L_3$ and $L_1 + L_2 - L_3$ lying along the edge of the Weyl chamber, as well as one with highest weight $L_1 + L_2$ lying in a face. We can thus find irreducible representations with highest weight γ, if not for every γ in $\Lambda_W \cap \mathscr{W}$, at least for every weight γ in the intersection of $\mathscr{W}$ with a sublattice of index 2 in Λ_W.

§19.2. Representations of the Even Orthogonal Algebras

We will not examine any further representations of $\mathfrak{so}_6\mathbb{C}$ per se, leaving it as an exercise to do so (and to compare the results to the corresponding analysis for $\mathfrak{sl}_4\mathbb{C}$). Instead, we can now describe the general pattern for representations of the even orthogonal Lie algebras $\mathfrak{so}_{2n}\mathbb{C}$. The complete story will have to wait until the following lecture, since at present we cannot construct all the representations of $\mathfrak{so}_{2n}\mathbb{C}$ (as we have pointed out, we have been able to do so in the cases $n = 2$ and 3 studied so far only by virtue of isomorphisms with other Lie algebras; and there are no more such isomorphisms from this point on). We will nonetheless give as much of the picture as we can.

To begin with, recall that the weight lattice of $\mathfrak{so}_{2n}\mathbb{C}$ is generated by $L_1, \ldots, L_n$ together with the further vector $(L_1 + \cdots + L_n)/2$. The Weyl chamber, on the other hand, is the cone

$$\mathscr{W} = \{\sum a_i L_i \colon a_1 \geq a_2 \geq \cdots \geq \pm a_n\}.$$

Note that the Weyl chamber is a simplicial cone, with faces corresponding to the n planes $a_1 = a_2, \ldots, a_{n-1} = a_n$ and $a_{n-1} = -a_n$; the edges of the Weyl chamber are thus the rays generated by the vectors L_1, $L_1 + L_2, \ldots,$ $L_1 + \cdots + L_{n-2}$, $L_1 + \cdots + L_n$ and $L_1 + \cdots + L_{n-1} - L_n$ (note that $L_1 + \cdots + L_{n-1}$ is not on an edge of the Weyl chamber). We see from this that, as in every previous case, the intersection of the weight lattice with the closed Weyl cone is a free semigroup generated by fundamental weights, in this case the vectors $L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-2}$ and the vectors[1]

$$\alpha = (L_1 + \cdots + L_n)/2 \quad \text{and} \quad \beta = (L_1 + \cdots + L_{n-1} - L_n)/2.$$

As before, the obvious place to start to look for irreducible representations is among the exterior powers of the standard representation. This almost works: we have

Theorem 19.2. (i) *The exterior powers $\wedge^k V$ of the standard representation V of $\mathfrak{so}_{2n}\mathbb{C}$ are irreducible for $k = 1, 2, \ldots, n-1$;* and (ii) *The exterior power $\wedge^n V$ has exactly two irreducible factors.*

Proof. The proof will follow the same lines as that of the analogous theorem for the symplectic Lie algebras in Lecture 17; in particular, we will start by considering the restriction to the same subalgebra as in the case of $\mathfrak{sp}_{2n}\mathbb{C}$.

Recall that the group $\mathrm{Sp}_{2n}\mathbb{C} \subset \mathrm{SL}_{2n}\mathbb{C}$ of automorphisms preserving the skew form Q introduced in Lecture 16 contains the subgroup G of automorphisms of the space $V = \mathbb{C}^{2n}$ preserving the decomposition $V = \mathbb{C}\{e_1, \ldots, e_n\} \oplus \mathbb{C}\{e_{n+1}, \ldots, e_{2n}\}$, acting as an arbitrary automorphism on the first factor and as the inverse transpose of that automorphism on the second factor; in matrices

$$G = \left\{ \begin{pmatrix} X & 0 \\ 0 & {}^tX^{-1} \end{pmatrix}, X \in \mathrm{GL}_n\mathbb{C} \right\}.$$

In fact, the subgroup $\mathrm{SO}_{2n}\mathbb{C} \subset \mathrm{SL}_{2n}\mathbb{C}$ also contains the same subgroup; we have, correspondingly a subalgebra

[1] To conform to standard conventions, with simple roots $\alpha_i = L_i - L_{i+1}$ for $1 \leq i \leq n-1$, and $\alpha_n = L_{n-1} + L_n$, to have $\omega_i(H_{\alpha_j}) = \delta_{i,j}$, the fundamental weights ω_i should be put in the order: $\omega_i = L_1 + \cdots + L_i$ for $1 \leq i \leq n-2$, and

$$\omega_{n-1} = \beta = (L_1 + \cdots + L_{n-1} - L_n)/2, \qquad \omega_n = \alpha = (L_1 + \cdots + L_n)/2.$$

$$\mathfrak{s} = \left\{\begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix}, A \in \mathfrak{sl}_n\mathbb{C}\right\} \subset \mathfrak{so}_{2n}\mathbb{C}$$

isomorphic to $\mathfrak{sl}_n\mathbb{C}$.

Denote by W the standard representation of $\mathfrak{sl}_n\mathbb{C}$. As in the previous case, the restriction of the standard representation V of $\mathfrak{so}_{2n}\mathbb{C}$ to the subalgebra $\mathfrak{s}$ then splits

$$V = W \oplus W^*$$

into a direct sum of W and its dual; and we have, correspondingly,

$$\wedge^k V = \bigoplus_{a+b=k} (\wedge^a W \otimes \wedge^b W^*).$$

We also can say how each factor on the right-hand side of this expression decomposes as a representation of $\mathfrak{sl}_n\mathbb{C}$: we have contraction maps

$$\Psi_{a,b}\colon \wedge^a W \otimes \wedge^b W^* \to \wedge^{a-1} W \otimes \wedge^{b-1} W^*;$$

and the kernel of $\Psi_{a,b}$ is the irreducible representation $W^{(a,b)}$ with highest weight $2L_1 + \cdots + 2L_a + L_{a+1} + \cdots + L_{n-b}$. The restriction of $\wedge^k V$ to $\mathfrak{s}$ is thus given by

$$\wedge^k V = \bigoplus_{\substack{a+b\le k \\ a+b\equiv k(2)}} W^{(a,b)},$$

where the actual highest weight factor in the summand $W^{(a,b)} \subset \wedge^k V$ is the vector

$$\begin{aligned} w^{(a,b)} &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \wedge Q^{(k-a-b)/2} \\ &= e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n} \wedge \left(\textstyle\sum (e_i \wedge e_{n+i})\right)^{(k-a-b)/2}. \end{aligned}$$

Now, all the vectors $w^{(a,b)}$ have distinct weights; and it follows, as in Exercise 17.7, that *any highest weight vector for the action of* $\mathfrak{so}_{2n}\mathbb{C}$ *on* $\wedge^k V$ *will be a scalar multiple of one of the* $w^{(a,b)}$. It will thus suffice, in order to show that $\wedge^k V$ is irreducible as representation of $\mathfrak{so}_{2n}\mathbb{C}$ for $k < n$, to exhibit for each (a, b) with $a + b \le k < n$ other than $(k, 0)$ a positive root α such that the image $\mathfrak{g}_\alpha(w^{(a,b)}) \ne 0$. This is simplest in the case $a + b = k < n$ (so there is no factor of Q in $w^{(a,b)}$): just as in the case of $\mathfrak{sp}_{2n}\mathbb{C}$ we have

$$\begin{aligned} &Y_{a+1,n-b+1}(w^{(a,b)}) \\ &\quad = (E_{a+1,2n-b+1} - E_{n-b+1,n+a+1})(e_1 \wedge \cdots \wedge e_a \wedge e_{2n-b+1} \wedge \cdots \wedge e_{2n}) \\ &\quad = w^{(a+1,b-1)} \\ &\quad \ne 0 \end{aligned}$$

and $Y_{i,j}$ is the generator of the positive root space $\mathfrak{g}_{L_i+L_j}$.

In case $a + b < k < n$, we observe first that for any i and j